이학박사학위논문

ABJM Scattering Amplitudes & Duality Test in Higher-Dimensional SCFTs

ABJM 산란 진폭 & 고차원 초등각장이론의 이중성 확인

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Abstract

ABJM Scattering Amplitudes &

Duality Test in Higher-Dimensional SCFTs

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This dissertation consists of two parts. In the first part, I discuss about ABJM scattering amplitudes. I develop a new gauge choice for the orthogonal Grassmannian formulation of ABJM amplitudes called u-gauge. Using the u-gauge choice, I compute 8-point ABJM amplitude for a first time. I also derive an ABJM soft theorem via the recursion relation for ABJM amplitudes. In the second part, I study UV dualities between higher-dimensional SCFTs by comparing their supersymmetric indices. More concretely, I focus on the UV duality between 5d $\mathcal{N} = 1$ $SU(3)$ with $N_f = 10$ fundamental hypermultiplets and 5d $\mathcal{N} = 1$ $Sp(2)$ gauge theory with same number of hypermultiplets. Furthermore, I study the index of their conjectured UV SCFT which is 6d $\mathcal{N} = (1, 0)$ SCFT on a circle with $Sp(1)$ gauge symmetry and $N_f = 10$ fundamental hypermultiplets. Instantons play an important role in the duality, and string theory gives a nice way of the ADHM construction for the super-Yang-Mills theories.

Keywords: ABJM, Scattering amplitude, Instanton, D-brane, Index

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Contents

Chapter 1

Introduction

String theory, which assumes relativity and quantum mechanics for two dimensional objects, has been believed to give a correct description for the quantum theory of gravity. In quantization of two-dimensional strings, gravitons and gluons appear very naturally. This is not the end of the story. While studying string theory, string theorists have found lots of dualities in string and gauge theories or discovered new physical objects like D-branes, M-theory branes. Using D-branes, M-theory branes and dualities, string theorists have made tremendous quantum field theories in various dimensions. For this reason, string theory is not only good for quantum gravity but also good for studying new quantum field theories.

When a theory interacts weakly like QED, one can study such quantum field theory by perturbation theory. One can compute tree-level or loop-level scattering amplitudes, which are used to compute scattering cross section or decay rate. These results are compared to real experimental data, and then one needs higher-order perturbations for precision test. For this reason, scattering amplitudes are considered as the most fundamental physical quantity. But for strongly interacting theories like Yang-Mills theory in low energy, perturbation theory is not valid anymore. There is no systematic tools to study such theories yet. One can sometimes use computer simulation like lattice QCD. Nonetheless, string theory sometimes gives a nice effective description for the strongly coupled system, and one can exactly compute some physical observables like supersymmetric index. For instance, strongly coupled 5d superconformal theory can be realized by (p, q)-web diagram, and it gives weakly interacting low energy effective 5d SYM description. In this case, instanton soliton particles play an important role in the strongly coupled regime. Therefore, instanton correction should be considered when computing a supersymmetric index of the theory. In conclusion, I want to emphasize that both perturbative physics and non-perturbative physics are really important in studying quantum field theories. In this dissertation, I will study the perturbative physics in 3d ABJM theory and non-perturbative physics in 5d gauge theories.

On ABJM scattering amplitudes

If a theory has Lagrangian with small coupling constant, one can use perturbation theory. Scattering amplitudes are one of the most fundamental quantity in quantum field theory [1]. Yang-Mills theory(or QCD) scattering amplitudes have been studied for a long time, because it is necessary for QCD precision test in collider experiment or because of theoretical interest it has. In Feynman diagram approach, the number of gluon-gluon scattering diagrams increases much more rapidly as the number of external gluons increases. For instance, $g + g \to 3g$ has 25 diagrams and $g + g \to 4g$ has 220 diagrams. The interesting thing is that after cumbersome calculation the amplitudes are organized in a simple form if one adopt fancy notation for scattering amplitude so-called *Spinor helicity formalism*. For instance, n-point MHV[1] gluon amplitudes are written as one line of equation so-called Parke-Taylor amplitudes [2]. After discovery of such simplification, scattering amplitudes have been studied in a different approach with Feynman diagram. One of the great triumph in scattering amplitudes is BCFW recursion relation [3]. One can construct arbitrary higher-point gluon scattering amplitudes by gluing lower-point amplitudes via the BCFW recursion relation. The BCFW recursion can be generalized to arbitrary high dimensions $d \geq 4$ and can be extended to other gauge theories including gravity, $\mathcal{N} = 4$ SYM, etc. [4, 5]. But 3d recursion relation operates rather differently from higher-dimensional one and will be discussed in more detail later [6].

One can extend pure Yang-Mills theory to $\mathcal{N} = 4$ super-Yang-Mills theory whose scattering amplitudes contain pure Yang-Mills one. $\mathcal{N} = 4$ SYM theory not only enjoys a Poincaré symmetry but also superconformal symmetry. Witten's twistor string has made a breakthrough on the scattering amplitude of $\mathcal{N} = 4$ SYM [7], and it leads to the discovery of RSV formula for $\mathcal{N} = 4$ SYM superamplitudes [8]. Such development changes the idea of scattering amplitude

[1]MHV(Maximally Helicity Violating) amplitude is the first non-trivial amplitude whose helicity sum is nonzero. Its helicity configuration is $(g^- g^- g^+ g^+ \dots g^+)$ when one take all gluons outgoing. If they contain all positive helicity gluons $(g^+ g^+ g^+ \dots g^+)$ or only one negative helicity gluons $(g^- g^+ g^+ \dots g^+)$, such amplitudes vanish.

computation, and it leads to the discovery of Grassmannian formula for $\mathcal{N}=4$ SYM amplitudes [9]. The Grassmannian formulation of $\mathcal{N}=4$ SYM scattering amplitudes reveals manifestly dual superconformal invariance as well as superconformal invariance [10], and finally it shows Yangian invariance [11]. The Grassmannian formula can be constructed by on-shell diagram [12], so Grassmannian can be understood as a symmetric representation of multi-BCFW recursion. Geometrical understanding of scattering amplitudes now reaches to the geometrical object called Amplituhedron [13].

As another area in the scattering amplitude research, soft behaviors of scattering amplitudes receive a lot of attention recently. Because it can be regarded as the key of black hole information paradox [14, 15]. Soft limit of scattering amplitudes is pioneered by Weinberg soft theorem [16]. Recently, sub-leading and sub-sub-leading parts of the graviton soft theorem were obtained in [17], and sub-leading soft gluon theorem was obtained in [18]. The soft theorems can be extended to arbitrary dimensions [19]. But in 3d gravity, gravitons have no propagating on-shell degrees of freedom because of its topological nature. Therefore, non-trivial amplitudes only exist when 3d gravity couples to matters. In this case, the 3d gravity theory can only have even-point scattering amplitudes of matters, and then one should think a double soft limit of matters [20]. The story is same for 3d Chern-Simon theory, which will be explained in detail later. Soft theorems show IR divergence of scattering amplitudes for massless theories, and they have an universal form for a given theory. For this reason, soft theorem can be use to verify whether an amplitude computation is correct or not.

ABJM theory which is a 3d $\mathcal{N} = 6$ Chern-Simons matter theory [21] is a close sibling of 4d $\mathcal{N} = 4$ SYM. Therefore, it is natural to think about properties of scattering amplitudes in ABJM theory. Many other common features have been found in ABJM scattering amplitudes. For instance, they admit a 3d version BCFW recursion relation [6] and reveal dual-superconformal invariance [6,22,23]. They also have Grassmannian formulation [24] and twistor string formulation [25–27] and on-shell diagram [28–30]. Even such a great success on ABJM amplitudes, explicit results for ABJM tree amplitudes to date are limited to 4- and 6-point amplitudes and few component of 8-point amplitude.

The goal of this study is to gather more data on ABJM amplitudes and to better understand ABJM amplitudes. In this work, I compute supersymmetric 8-point amplitude for a first time and derive a double soft theorem for tree amplitudes. To compute the 8-point amplitude, I develop a new gauge choice for the orthogonal Grassmannian called U-gauge. This gauge choice seems good for not only 8-point but also higher-point computations. For technical reason, my results are limited on 4-,6-,8-point amplitudes. I also derive a double soft theorem for the ABJM theory. The proof is based on the ABJM recursion relation [6]. I obtain leading and sub-leading parts in a soft factor. The double soft theorem is used to confirm my 8-point result. This work is based on my paper in collaboration with Chin and Lee [31].

On 5d quantum field theories

There are many kinds of duality in theoretical physics such as T-duality, S-duality. Sometimes duality framework sheds light on new physics like the discovery of M-theory. By studying dualities between five different string theories, the existence of a hidden circle in type IIA string theory had been found, and it led to the discovery of M-theory [32,33]. Seiberg duality is an famous example of duality in the gauge theory, which is a 4d $\mathcal{N} = 1$ supersymmetric version of the EM duality [34]. Seiberg duality says two different gauge theories are dual in IR limit. In the IR limit, one theory is strongly interacting but the other theory is weakly interacting. Therefore, one can study a strongly interacting system by studying its weakly coupled dual theory. One way of realizing Seiberg duality in string theory is D-brane realizations of the gauge theory [35]. Open strings can end on the N Dp-branes, and low energy effective theory of the system is described by $U(N)$ supersymmetric Yang-Mills theory in $(p+1)$-dimensions. One can obtain other kinds of gauge group ($Sp(N)$ or $SO(N)$) or can couple matters by introducing Orientifold planes or other branes. Using the string theory brane configurations and brane movements, one can realize Seiberg duality [36]. D-brane engineered gauge theory is useful not only for lower-dimensional theories $d \leq 4$ but also for higher dimensional one $d > 4$.

It was first shown by Nahm that the highest dimensions for superconformal field theory can exist is $d = 6$ [37]. Existence for non-trivial UV-fixed point of five dimensional $\mathcal{N} = 1$ gauge theories was first claimed by Seiberg in 1996 by computing 1-loop quantum prepotential [38]. After that, possible 5d gauge theories are classified by Intriligator, Morrison and Seiberg(IMS) by inspecting effective gauge couplings in entire Coulomb branch moduli space [39] . IMS said that 5d gauge theory only can have simple gauge group, and representation and the number of matters are restricted. This statement is called IMS bound. For instance, $SU(2)$ gauge theory only can have $N_f \leq 7$ fundamental hypermultiplets. Some of these higher-dimensional superconformal field theories can be realized by brane systems, which gives effective SYM description. For instance, (p, q)-web diagrams represent 5d SYM descriptions for 5d SCFTs [40–42]. Higher-dimensional($d > 4$) gauge theory is non-renormalizable at perturbative level, so UV-incomplete. But in the strong coupling limit instanton particles become massless since instanton mass proportional to the inverse gauge coupling $m_{\text{inst}} \sim 1/g_5^2$. So lightest instanton particles are playing an important role in the strong coupling regime.

6d $\mathcal{N} = (2, 0)$ SCFTs are classified by ADE, which is obtained by stack of M5-branes in ADE singularities [43]. Recently 6d $\mathcal{N} = (1, 0)$ SCFTs classification was proposed in [44–46]. 6d SCFTs contain tensor multiplets, and they couples to tensionless self-dual strings. 6d SCFTs usually have been studied in a tensor branch where tensor multiplet scalar has non-vanishing VEV. On the tensor branch, they admit effective 6d SYM descriptions, and inverse gauge coupling is proportional to the tensor scalar VEV. Compactifications of 6d SCFTs lead to non-trivial dualities in lower-dimensional theories, e.g. [47]. Studying higher-dimensional theory can gives new insight on lower-dimensional theories and new dualities. This is a quite nice motivation for studying higher-dimensional gauge theories.

Main interest of this thesis is UV duality in 5d gauge theories and its UV SCFTs. Under the Seiberg's guidance, 5d physics had been studied for a long time. But recent observation gives the evidence for theories beyond IMS bound [48–58]. The main topic of the thesis is studying 5d gauge theories beyond IMS bound. Very recent work on the 5d gauge theories gave a beautiful physical interpretation of how theories can exist beyond IMS bound [59]. I will focus on two specific 5d gauge theories. One has $Sp(N+1)$ gauge group and another one has $SU(N+2)$ gauge group. Both theories are coupled to $N_f = 2N+8$ fundamental hypermultiplets. The claim made in [56] is that above two 5d gauge theories have same UV-fixed point, and this UV SCFT is not a 5d SCFT but a 6d SCFT compactified on a circle. The conjectured 6d UV SCFT has $\mathcal{N} = (1,0)$ SUSY and $Sp(N)$ gauge group with $N_f = 2N+8$ hypermultiplets. Motivation of this work is a quantitative test of this duality between 5d-6d by comparing their BPS indices. This result based on my own work for 5d $Sp(2)$(and $Sp(N+1)$ generalization) gauge group [60] and unpublished work for 5d $SU(3)$ gauge group. I will check the simplest non-trivial duality i.e. $N = 1$ case($N = 0$ is trivial because $Sp(1) \cong SU(2)$). Instantons are playing an important role in the duality, and the instanton contribution in the indices is the main subject of the thesis. This kind of 5d-6d duality is not a novel feature. Maximal 5d SYM also has shown this feature [61, 62], and 5d SYM description of a circle reduced 6d E-string theory [63] also has shown this feature [64].

To test this duality, I compute Nekrasov partition functions which count BPS bound states of the theories [65–67], and instantons form marginal bound states with W-bosons and their superpartners. Therefore, one needs to study instantons to compute Nekrasov partition functions. To study instantons, I use UV-complete stringy ADHM gauged quantum mechanics which is realized by D-brane systems instead of UV-incomplete ADHM instanton quantum mechanics. However, since the stringy ADHM construction embeds the instanton quantum mechanics into string theory, it often contains unwanted extra degrees of freedom which are not included in the QFT that one is interested in. So when one compute instanton partition functions via the stringy ADHM partition functions, the extra contribution part should be subtracted to obtain correct QFT instanton partition functions [64]. One could separately compute these extra contributions from string theory considerations [64]. Unfortunately, it is not known how to compute these extra contributions for the 5d $Sp(N+1)$ and $SU(N+2)$ gauge theories with $N_f = 2N+8$ hypermultiplets. This will be explained more in Chapter 3. Nevertheless, in the 5d $Sp(2)$ and $SU(3)$ examples, one can detour this problem with simple tricks. I will explain the tricks in Chapter 3. Using such tricks, Nekrasov partition functions of two theories can be computed. Then, I compare their indices under the appropriate fugacity map and confirm the duality.

If one consider duality between 5d and 6d, part of 5d W-bosons can be regarded as 6d self-dual strings wrapping the circle, and instantons can be regarded as KK momenta on these strings like E-string [64]. So one can study the same physics from the elliptic genera of 6d self-dual strings. I compute these elliptic genera of the conjectured 6d UV SCFT and compare them with instanton partition function of the 5d gauge theories. I find a perfect agreement of two indices under the appropriate fugacity map, which provide non-trivial supports of the proposal made in [55]. In particular, my test clarifies the physical setting of the 5d-6d dualities, by emphasizing the roles of background Wilson lines, and also by explicitly showing the relations between various 5d and 6d parameters.

Organization of the thesis

The thesis is organized as follows. In the Chapter 2, I focus on ABJM scattering amplitudes. At first, I will briefly review very basics on ABJM amplitudes like momentum-spinors, on-shell superfields, etc. Then I will introduce modern technology on scattering amplitudes called Grassmannian(or orthogonal Grassmannian). ABJM amplitudes are given by the orthogonal Grassmannian(OG) formula. I will introduce a new $GL(k)$ fixing condition in OG_{2k} called U-gauge. This gauge choice is very useful for higher-point amplitude computations. I reproduce known 4- and 6-point amplitudes very easily via u-gauge. Then I will show 8-point amplitude computation. Finally, I will introduce the ABJM soft theorem and prove it via the recursion relation for ABJM theory. This soft theorem is used to confirm the my 8-point result.

In the Chapter 3, I focus on higher-dimensional quantum field theories. I will review on the old story about 5d SCFTs based on the Seiberg's argument, and briefly mention about the recent claim on beyond Intriligator-Morrison-Seiberg(IMS) bound. I'm going to compute Nekrasov partition functions of very specific gauge theories which I already mentioned above. Instanton partition functions can be computed by Witten index of the stringy ADHM gauged QM, but it does not work with too many hypermultiplets. I will show how I can overcome this problem, then I will compare two indices of the 5d theories with highly non-trivial fugacity made in [57]. Finally, I will study a BPS index of the conjecture 6d UV SCFT. Using the brane diagram for this theory on the tensor branch, I compute elliptic genera of the instanton soliton strings in the effective 6d gauge theory. By comparing the 6d index with the 5d index, I confirm the 5d-6d duality.

Chapter 2

ABJM Scattering Amplitudes

2.1 ABJM amplitudes

In this section, I will discuss about the elementary story on scattering amplitudes of 3d $\mathcal{N} = 6$ Chern-Simons matter theory called ABJM theory. The 3d ABJM theory is very closed friend of the 4d $\mathcal{N} = 4$ super-Yang-Mills theory. Therefore, it is natural to apply the modern technology on 4d scattering amplitudes to ABJM theory amplitudes.

2.1.1 ABJM theory

ABJM theory is known as a world volume theory of multiple M2-branes [21]. ABJM theory is 3-dimensional Chern-Simons matter theory with gauge group $U(N)_k \times U(N)_{-k}$ and four complex scalars and four fermions with their complex conjugates. The scalar fields and the fermion fields have bi-fundamental representation under the two gauge groups, so they are denoted by $(\phi_A)^{\dot{a}}{}_{a}$ and

$(\psi^A)^{\dot{a}}{}_a$. The indices ($\dot{a} = 1, \ldots, N$), ($a = 1, \ldots, N$) and ($A = 1, 2, 3, 4$) are the $U(N)_k$, $U(N)_{-k}$ and $SO(6)_R \cong SU(4)_R$ symmetry indices. The Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{ABJM}} = \frac{k}{2\pi}\Big[\frac{1}{2}\epsilon_{\mu\nu\rho}\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho - \hat{A}_\mu\partial_\nu\hat{A}_\rho - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho\right) \\ - (D^\mu\phi_A)^\dagger(D^\mu\phi_A) + i\bar{\psi}_A \not{D}\psi^A + \mathcal{L}_4 + \mathcal{L}_6\Big]\,. \qquad (2.1)\end{aligned}$$

The covariant derivatives for the bi-fundamental fields are

$$D_\mu\phi_A \equiv \partial_\mu\phi_A + i\hat{A}_\mu\phi_A - i\phi_A A_\mu\,, \qquad (2.2)$$

$$(D_\mu\phi_A)^\dagger \equiv \partial_\mu\bar{\phi}^A + iA_\mu\bar{\phi}^A - i\bar{\phi}^A\hat{A}_\mu\,, \qquad (2.3)$$

and same for ψ^A and $\bar{\psi}_A$. The quartic and sextic interaction terms are given by

$$\begin{aligned}\mathcal{L}_4 = i\ \mathrm{Tr}\Big(&\bar{\phi}^B\phi_B\bar{\psi}_A\psi^A - \phi_B\bar{\phi}^B\psi^A\bar{\psi}_A + 2\phi_A\bar{\phi}^B\psi^A\bar{\psi}_B - 2\bar{\phi}^A\phi_B\bar{\psi}_A\psi^B \\ &- \epsilon_{ABCD}\bar{\phi}^A\psi^B\bar{\phi}^C\psi^D + \epsilon^{ABCD}\phi_A\bar{\psi}_B\phi_C\bar{\psi}_D\Big)\,, \qquad (2.4)\\ \mathcal{L}_6 = \frac{1}{3}\ \mathrm{Tr}\Big(&\phi_A\bar{\phi}^A\phi_B\bar{\phi}^B\phi_C\bar{\phi}^C + \bar{\phi}^A\phi_A\bar{\phi}^B\phi_B\bar{\phi}^C\phi_C + 4\bar{\phi}^A\phi_B\bar{\phi}^C\phi_A\bar{\phi}^B\phi_C \\ &- 6\phi_A\bar{\phi}^B\phi_B\bar{\phi}^A\phi_C\bar{\phi}^C\Big)\,. \qquad (2.5)\end{aligned}$$

Using the Lagrangian one can easily read the Feynman rule under the appropriate gauge fixing condition [6], but my interest is not Feynman diagrammatic computation for scattering amplitudes but new technology for scattering amplitudes. This new technology will be introduced in below.

Note that the dynamics of 3d Chern-Simons theory is different with 3d Yang-Mills theory. The equation of motion for the pure Chern-Simons action is given by

$$\partial_{[\mu}A_{\nu]} + i[A_\mu, A_\nu] = F_{\mu\nu} = 0\,, \qquad (2.6)$$

and the solution of the equation is just pure gauge $A_\mu = g\partial_\mu g^{-1}$, where g is an arbitrary group element of $U(N)$. One can choose a gauge such that $A_\mu = 0$, so this implies that external Chern-Simons gauge bosons do not have any physical degrees of freedom. Therefore, one have to study scattering of matters, because gauge bosons have no dynamical degrees to scatter. If scattering amplitudes contain at least one Chern-Simons gauge boson in external legs, it trivially vanishes. This implies that only even-point scattering amplitudes are non-trivial in ABJM theory. Otherwise, they should contain at least one external gauge boson, and it leads to vanishing results.

2.1.2 ABJM scattering amplitudes

Momentum spinors in 3d

When one study scattering amplitudes of massless gluons in 4d, *spinor helicity formalism* have been used instead of ordinary momentum vectors and polarization vectors. Similarly, one can define 3d momentum bi-spinor by taking $p^{\mu=2} = 0$ in 4d momentum p^μ

$$p_{ab} = p^\mu(\sigma_\mu) = \begin{pmatrix} -p^0 + p^3 & p^1 \\ p^1 & -p^0 - p^3 \end{pmatrix}, \quad \text{and} \quad \det(p) = -p_\mu p^\mu = m^2 \,. \tag{2.7}$$

For massless theory, this implies momentum bi-spinor can be rewritten as a product of two component commuting spinor variable

$$p^{ab} = \lambda^a \lambda^b \,. \tag{2.8}$$

The λ^a should be real or purely imaginary for reality condition of p^{ab}, and such λ^α is real (purely imaginary) for outgoing (incoming) particles. I will use the bracket notation for momentum spinors $\lambda_i^a = |i\rangle^a$. The spinor index a can be raised or lowered by Levi-Civita symbol of the $SL(2,\mathbb{R}) = \text{Spin(1,3)}$. All Lorentz invariants are given by inner product of two momentum spinors with Levi-Civita symbol

$$\langle ij\rangle \equiv \lambda_i^a \epsilon_{ab} \lambda_j^b = -\langle ji\rangle \,. \tag{2.9}$$

Maldelstam variables such as $s = -(p_1+p_2)^2$ is given by

$$s_{ij} = -(p_i+p_j)^2 = \langle ij\rangle^2 \,. \tag{2.10}$$

One can also think about little group for 3d massless kinematics, which is $\mathbb{Z}_2$. The little group $\mathbb{Z}_2$ acts on spinors as $\lambda^a \to -\lambda^a$, and it is called λ-parity. Under the λ-parity, 3d scalar particles scale with $(+1)$ and 3d fermions scale with (-1). One can also find Schouten's identity

$$\langle kl\rangle\langle ij\rangle + \langle ki\rangle\langle jl\rangle + \langle kj\rangle\langle li\rangle = 0 \,, \tag{2.11}$$

and it will be useful in amplitude computations.

On-shell superspace & Superconformal algebra

The minimal spinor in 3d is a two component Majorana spinor, which satisfy Majorana reality condition. ABJM theory has $\mathcal{N} = 6$ supersymmetry, and R-symmetry is $SO(6)$. Six real supercharges can be combined into 3 complex supercharges Q^{aA} and their complex conjugate $\tilde{Q}^a_A$, where $A = 1,2,3$ is reduced $SU(3)_R$ symmetry index. One can introduce on-shell superspace coordinates η_{iA} for external matters, and then the supercharges can be realized by

$$Q^{aA} = \sum_{i=1}^{n} q_i^{aA} = \sum_{i=1}^{n} |i\rangle^a \frac{\partial}{\partial \eta_{iA}} \,, \qquad \tilde{Q}^a_A = \sum_{i=1}^{n} \tilde{q}^a_{iA} = \sum_{i=1}^{n} |i\rangle^a \eta_{iA} \,. \tag{2.12}$$

ABJM theory not only has super-Poincaré symmetry but also has bigger symmetry known as superconformal symmetry : $OSp(6|4)$. Superconformal generators consist of 24 bosonic generators and 24 fermionic generators, and they are given by

$$\begin{array}{ccc}
P^{ab} = \sum_i |i\rangle^a |i\rangle^b & M_{ab} = \sum_i \langle i|_{(a}\partial_{|i\rangle^{b)}} & Q^{aA} = \sum_{i=1} |i\rangle^a \frac{\partial}{\partial\eta_{iA}} \\
K_{ab} = \sum_i \partial_{|i\rangle^a}\partial_{|i\rangle^b} & D = \sum_i (\frac{1}{2}|i\rangle_{(a}\partial_{|i\rangle^{a)}} + \frac{1}{2}) & \tilde{Q}^a_A = \sum_{i=1} |i\rangle^a \eta_{iA} \\
R_{AB} = \sum_i \eta_{iA}\eta_{iB} & R_A{}^B = \sum_i (\eta_{iA}\partial\eta_{iB} - \frac{1}{2}\delta^B_A) & S_{aA} = \sum_i \partial_{|i\rangle^a}\eta_{iA} \\
R^{AB} = \sum_i \partial_{\eta_{iA}}\partial_{\eta_{iB}} & & \tilde{S}^A_a = \sum_i \partial_{|i\rangle^a}\partial\eta_{iA} \, .
\end{array} \tag{2.13}$$

Scattering amplitudes should be invariant under the such superconformal transformation, and it leads to the discovery of the Grassmannian formulation of scattering amplitudes. I will discuss about the Grassmannian later.

Color-ordered amplitude

ABJM theory has 4 complex scalars and 4 complex fermions and their complex conjugates, and they transform fundamental or anti-fundamental in the original $SU(4)_R$ symmetry group. These matter fields can be combined into on-shell superfield

$$\Phi = \phi_4 + \eta_A\psi^A - \frac{1}{2}\epsilon^{ABC}\eta_A\eta_B\phi_C - \eta_1\eta_2\eta_3\psi^4 \, , \tag{2.14}$$

$$\bar{\Psi} = \bar{\psi}_4 + \eta_A\bar{\phi}^A - \frac{1}{2}\epsilon^{ABC}\eta_A\eta_B\bar{\psi}_C - \eta_1\eta_2\eta_3\bar{\phi}^4 \, , \tag{2.15}$$

where only $SU(3)$ R-symmetry is manifest. Since only matter fields have dynamical degrees, one should consider scattering of matters. If you consider Feynman rules for ABJM theory, the number of external legs should be even as I explained in introduction. For odd number of external legs, there should be at least one Chern-Simons gauge boson in external legs, and the amplitude has to vanish.

One can also check this by considering $U(1)$ part of the R-symmetry generator $R_C{}^C = \sum_i (\eta_{iC}\partial\eta_{iC} - \frac{3}{2})$. If this act on n-point ABJM scattering amplitudes, it should be vanish $R_C{}^C \mathcal{A}_n = 0$. This condition gives following result

$$\sum_i \eta_{iC}\partial\eta_{iC} = n\frac{3}{2}\mathcal{A}_n\,. \tag{2.16}$$

But scattering amplitudes can't have fractional degree of η_{iA}. It implies n should be even. This is a common feature of 3d superconformal theories with $\mathcal{N}=$ even. If one can consider Yang-Mills theory in 3d, they can have odd external legs, because this theory is not superconformal.

The matter superfields defined in (2.14) and (2.15) transform in the $(N,\bar{N})$ and $(\bar{N},N)$ representation under the $U(N)\times U(N)$ gauge group. One can define the $n=2k$-point color-ordered ABJM amplitude $\mathcal{A}_{2k}$ from the ordinary ABJM scattering amplitude $\hat{\mathcal{A}}_{2k}$. Definition of the color-ordered ABJM amplitude is as follows

$$\begin{aligned}&\hat{\mathcal{A}}_{2k}\left(\bar{\Psi}_1{}^{a_1}{}_{\dot{a}_1}, \Phi_2{}^{\dot{b}_2}{}_{b_2}, \bar{\Psi}_3{}^{a_3}{}_{\dot{a}_3}, \cdots, \Phi_{2k}{}^{\dot{b}_{2k}}{}_{b_{2k}}\right)\\ &\quad= \sum_{\sigma\in S_k, \bar{\sigma}\in S_{k-1}} \mathcal{A}_{2k}(\bar{1},\sigma_1,\bar{\sigma}_1,\ldots,\bar{\sigma}_{k-1},\sigma_k) \times \delta^{\dot{b}_{\sigma_1}}_{\dot{a}_{\bar{1}}}\cdots\delta^{\dot{b}_{\sigma_k}}_{\dot{a}_{\bar{\sigma}_{k-1}}} \times \delta^{b_{\bar{\sigma}_1}}_{a_{\sigma_1}}\cdots\delta^{b_{\bar{\sigma}_1}}_{a_{\sigma_k}}\,,\end{aligned} \tag{2.17}$$

and the color-ordered amplitude is invariant under the two site cyclic rotation up to sign

$$\mathcal{A}_{2k}(\bar{1},2,\bar{3},\cdots,2k) = (-1)^{k-1}\mathcal{A}_{2k}(\bar{3},4,\cdots,2k,\bar{1},2)\,. \tag{2.18}$$

I will use following collective notation $\Lambda_i = (\lambda_i,\eta_i)$ for external particles. Then the scattering amplitude is rewritten as

$$\mathcal{A}_{2k} = \mathcal{A}_{2k}(\Lambda_1,\Lambda_2,\ldots,\Lambda_{2k})\,. \tag{2.19}$$

Under the λ-parity $\lambda \to -\lambda$, $\eta_A \to -\eta_A$, fermionic wave functions pick up a minus sign. This implies that

$$\mathcal{A}_{2k}(\Lambda_1, \ldots, -\Lambda_i, \ldots, \Lambda_{2k}) = (-1)^i \mathcal{A}_{2k}(\Lambda_1, \ldots, -\Lambda_i, \ldots, \Lambda_{2k}) \,. \tag{2.20}$$

So far I have introduced the basics of ABJM scattering amplitudes. Color-ordered superamplitudes or shortly amplitudes should be invariant under the superconformal transformation. Furthermore, the amplitudes are not only invariant under the superconformal transformation but also the dual superconformal transformation [6,22,23], and it leads to the discovery of Yangian invariant formulation of ABJM scattering amplitudes [24]. Before discussing about the Yangian invariant formulation of ABJM amplitudes, let us discuss about the recursion relation for 3d theories.

2.2 3d recursion relation

Unlike 4d or higher-dimensions, it is not easy to derive the 3d recursion relation, since they do not admit the usual momentum shift

$$\begin{aligned}
&1)\ \hat{p}_i(z)^\mu = p_i^\mu - zq^\mu \,, \quad \hat{p}_j(z)^\mu = p_j^\mu + zq^\mu \,, \\
&2)\ p_i \cdot q = p_j \cdot q = q^2 = 0 \,,
\end{aligned}$$

i.e. $q^\mu = 0$ in three dimensions. Because of this fact, the 3d recursion relation was obtained by shifting momentum spinors rather different way in 4d [6].

If one take following BCFW shift for 3d momentum spinors

$$\begin{pmatrix} \hat{\lambda}_i(z) \\ \hat{\lambda}_j(z) \end{pmatrix} = R(z) \begin{pmatrix} \lambda_i \\ \lambda_j \end{pmatrix} \,, \tag{2.21}$$

the matrix $R(z)$ should be orthogonal matrix to respect momentum conservation

$$\hat{p}_i^{\alpha\beta}(z) + \hat{p}_j^{\alpha\beta}(z) = \begin{pmatrix} \hat{\lambda}_i(z)^\alpha & \hat{\lambda}_j(z)^\alpha \end{pmatrix} \begin{pmatrix} \hat{\lambda}_i(z)^\beta \\ \hat{\lambda}_j(z)^\beta \end{pmatrix} = \begin{pmatrix} \lambda_i^\alpha & \lambda_j^\alpha \end{pmatrix} \begin{pmatrix} \lambda_i^\beta \\ \lambda_j^\beta \end{pmatrix} = p_i^{\alpha\beta} + p_j^{\alpha\beta}$$

$$\Rightarrow \quad R(z)^T \cdot R(z) = I\,. \tag{2.22}$$

I use following parametrization for the orthogonal matrix $R(z)$

$$R(z) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{z+z^{-1}}{2} & -\frac{z-z^{-1}}{2i} \\ \frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{pmatrix}\,. \tag{2.23}$$

On-shell superspace coordinates $\eta_{iA}\,, \eta_{jB}$ are also shifted in the same way of momentum spinors (2.21). One can recover un-shifted variables by taking $z = 1$, i.e. $R(1) = 1$. Finally, one can obtain the contour integral formula for ordinary amplitudes like 4d BCFW

$$\mathcal{A}_n = \frac{1}{2\pi i} \oint_{z=1} \frac{\mathcal{A}_n(z)}{z-1}\,, \tag{2.24}$$

where the contour is chosen only picks the pole at $z = 1$.

If $\mathcal{A}_n(z)$ goes to zero as $z \to \infty$, likewise 4d BCFW, the contour integral formula (2.24) becomes sum of residues at finite $z \neq 0, 1$ by deforming the contour to infinity. Without loss of generality, one can take BCFW shift for 1 and n particles. Under the such shift, poles and residues are coming from the factorization diagrams like Figure 2.1. Deformed momentum bi-spinors are given by

$$\begin{aligned} \hat{p}_1^{\alpha\beta}(z) &= \frac{1}{2}(p_1^{\alpha\beta} + p_n^{\alpha\beta}) + z^2 q^{\alpha\beta} - z^{-2}\tilde{q}^{\alpha\beta}\,, \\ \hat{p}_n^{\alpha\beta}(z) &= \frac{1}{2}(p_1^{\alpha\beta} + p_n^{\alpha\beta}) - z^2 q^{\alpha\beta} + z^{-2}\tilde{q}^{\alpha\beta}\,, \end{aligned} \tag{2.25}$$

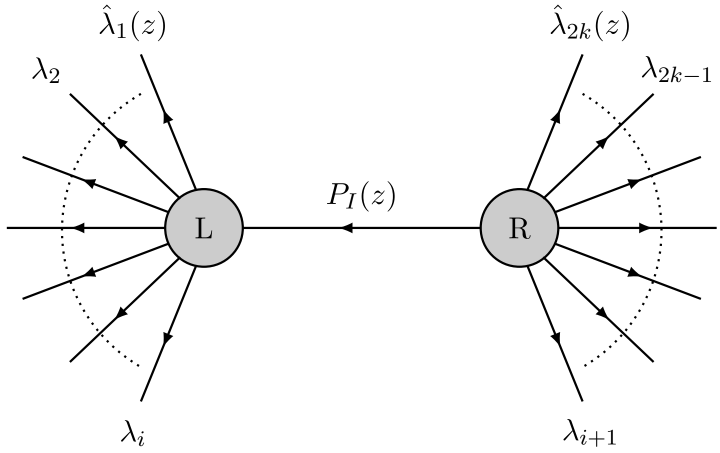


Figure 2.1 Factorization diagram for 1 and n BCFW shift.

where shifted parts q and $\tilde{q}$ are given by

$$\begin{aligned} q^{\alpha\beta} &= \frac{1}{4}(|1\rangle + i|n\rangle)^\alpha(|1\rangle + i|n\rangle)^\beta\,, \\ \tilde{q}^{\alpha\beta} &= \frac{1}{4}(|1\rangle - i|n\rangle)^\alpha(|1\rangle - i|n\rangle)^\beta\,. \end{aligned} \tag{2.26}$$

With the shifted momenta, on-shell conditions of the internal propagator become

$$\begin{aligned} P_I^2(z) &= (\hat{p}_1(z) + p_2 + \cdots + p_i)^2 \\ &= \langle\tilde{q}|P_{23\ldots i}|\tilde{q}\rangle z^{-2} + \langle q|P_{23\ldots i}|q\rangle z^2 - (P_{23\ldots i}\cdot P_{i+1\ldots n-1}) = 0\,, \end{aligned} \tag{2.27}$$

where I used collective notation for the internal momenta $P_{12\ldots i} = p_1 + p_2 + \cdots + p_i$. The solutions of the on-shell conditions are given by

$$\{(z_+)^2, (z_-)^2\} = \frac{(P_{23\ldots i}\cdot P_{i+1\ldots n-1}) \pm \sqrt{(P_{23\ldots i})^2 (P_{i+1\ldots n-1})^2}}{2\langle q|P_{2\ldots i}|q\rangle}\,. \tag{2.28}$$

So four poles appear for each diagram. After sum over the all poles in each diagram, and sum over all diagrams, the recursion relation is reduced to

$$\mathcal{A}_n = \sum_I \int d^3\eta_I \left(\mathcal{A}_L(z_+; i\eta_I)\frac{H(z_+, z_-)}{P_I^2(0)}\mathcal{A}_R(z_+, \eta_I) + (z_+ \leftrightarrow z_-)\right)\,, \tag{2.29}$$

where BCFW kernel $H(x,y)$ is given by

$$H(x,y)=\frac{x^2(y^2-1)}{x^2-y^2}\,. \tag{2.30}$$

Here I used the identity

$$\big((z_+^2-1)(z_-^2-1)\big)=\frac{P_{12\ldots i}^2}{\langle q|P_{2\ldots i}|q\rangle}\,, \tag{2.31}$$

and the integral of η_I comes from the summation of all internal states. Imaginary number i in $\mathcal{A}_L$ comes form the analytic continuation of the incoming particles to outgoing particles $|-p\rangle\to i|p\rangle$ and $\eta_{-P}\to i\eta_P$. The 3d recursion holds for any 3d theories if they satisfy '$\mathcal{A}_n(z)$ goes to zero as $z\to\infty$' condition. Until now ABJM and BLG theories are known having such behavior.

In principle, one can construct arbitrary higher-point ABJM amplitudes using the 3d recursion relation. However, as one can see above, it contains many square roots in the middle of calculation, and such square roots must be canceled miraculously to give analytic form of scattering amplitudes. For this reason, only some component amplitudes are computed by the recursion relation until now. In the next section, I will discuss about the Grassmannian formulation of ABJM amplitudes, which is much nice computing tool for ABJM amplitudes.[1]

2.3 Grassmannian formulation

4d $\mathcal{N}=4$ SYM is superconformal, dual superconformal invariant and finally Yangian invariant [11]. The Grassmannian formulation was first studied to ob-

[1] Grassmannian formula can be understood as many recursion relation of 4-point on-shell amplitudes called on-shell diagram [28–30]. Empirically, it seems Grassmannian computation is much simpler than recursion relation.

tain manifestly Yangian invariant formulation of scattering amplitudes. In such formulation, locality and unitarity are just emergent phenomena. The Grassmannian formula for maximal SYM scattering amplitudes was first proposed by Arkani-Hamed, Cachazo, Cheung, and Kaplan [9], and the on-shell diagram origin of the Grassmannian was discovered later in [12].

ABJM theory is also known to enjoy superconformal as well as dual superconformal symmetry [22, 23]. Therefore, it is natural to think about the Grassmannian formulation of ABJM amplitudes.

Orthogonal Grassmannian

Grassmannian formulation of ABJM amplitudes was discovered by Sangmin Lee [24]. The lack of momentum twistors in 3d leads slightly different form of the Grassmannian formula. As a result, ABJM Grassmannian geometry has additional constraints in their equation. The $2k$-point ABJM amplitude is given by following Grassmannian formula

$$\mathcal{L}_{2k}(\Lambda_i) = \int \frac{d^{k\times 2k}C}{\text{vol}[GL(k)]} \frac{\delta^{k(k+1)/2}(C\cdot C^T)\delta^{2k|3k}(C\cdot\Lambda)}{M_1 M_2\cdots M_{k-1}M_k}\,, \tag{2.32}$$

where Λ_i denotes a collection of momentum-spinor and on-shell superspace coordinate $\Lambda_i = \{\lambda_i^a, \eta_i^A\}$ for a external matter. The integration variable C is a $(k\times 2k)$ matrix. The matrix dot products denote $(C\cdot C^T)_{mn} = C_{mi}C_{ni}$, $(C\cdot\Lambda)_m = C_{mi}\Lambda_i$. The consecutive minor M_i is defined by

$$M_i = \epsilon^{m_1\cdots m_k} C_{m_1(i)}C_{m_2(i+1)}\cdots C_{m_k(i+k-1)}. \tag{2.33}$$

The $(k\times 2k)$ matrix C appears in (2.32) is defined with the equivalence relation $C\sim gC$ ($g\in GL(k)$) and the orthogonality constraint $C\cdot C^T = 0$. This is called orthogonal Grassmannian(OG_{2k}) in mathematical literature. The dimensions of

OG_{2k} is given by

$$\dim_{\mathbb{C}}[\mathrm{OG}_{2k}] = 2k^2 - k^2 - \frac{k(k+1)}{2} = \frac{k(k-1)}{2}. \tag{2.34}$$

The Grassmannian formula shows manifest two cyclic symmetry that ABJM amplitudes have. One can show superconformal invariance of the Grassmannian formula (2.32) with following steps. The superconformal generators are classified by $\Lambda\frac{\partial}{\partial\Lambda}$, $\frac{\partial^2}{\partial\Lambda\partial\Lambda}$, and $\Lambda\Lambda$. Action of the linear derivative generators $\Lambda\frac{\partial}{\partial\Lambda}$ on (2.32) gives vanishing result by support of the delta functions $\delta^{2|3}(C\cdot\Lambda)$. Action of second derivative generators $\frac{\partial^2}{\partial\Lambda\partial\Lambda}$ on (2.32) gives vanishing result by support of the orthogonality condition of Grassmannian $\delta^{k(k+1)/2}(C\cdot C^T)$. One can show that $\Lambda\Lambda$ action gives vanishing result by considering orthogonal complement $\tilde{C}$ of C, which satisfies

$$\tilde{C}\tilde{C}^T = 0\,, \quad \tilde{C}\cdot C^T = I_{k\times k}\,. \tag{2.35}$$

Then completeness relation says $\Lambda^T\cdot\Lambda = \Lambda^T(C^T\tilde{C} + \tilde{C}^T C)\Lambda = 0$ by support of $\delta^{2|3}(C\cdot\Lambda)$. Yangian invariance of the formula was first argued in [24], and it was explicitly proven later in [68].

The Grassmannian formula (2.32) should be interpreted as a contour integral on the moduli space of matrix C. After integrating out the bosonic delta functions, the number of actual integral variables are given by

$$\dim_{\mathbb{C}}[\mathrm{OG}_{2k}] - (2k-3) = \frac{(k-2)(k-3)}{2}\,, \tag{2.36}$$

where last 3 comes from overall momentum conservation delta function. For 4-point and 6-point amplitudes, there is no contour integral. But after 8-point, one should do the contour integral with appropriate contour description which can be read from on-shell diagram construction of orthogonal Grassmannian [28].

Explicit results for ABJM tree amplitudes to date are limited to 4- and 6-point amplitudes and few component of 8-point amplitude. I just want to take a few steps toward ABJM amplitudes by calculating higher-point amplitudes. To compute higher-point amplitudes, one needs more systematic tools, and I develop called U-gauge.

2.3.1 U-gauge

U-gauge choice is motivated by an early work on the pure spinor which is mathematically equivalent with orthogonal Grassmannian [69].

Light-cone basis

It is more convenient to take a real slice of the complex orthogonal Grassmannian with the split signature, where the "metric" in the particle basis is

$$g = \mathrm{diag}(-,+,-,+,\cdots)\,. \tag{2.37}$$

In this basis, the momenta and kinematic invariants are given by

$$p_i^{\alpha\beta} = (-1)^i \lambda_i^\alpha \lambda_i^\beta\,, \quad \lambda_i^\alpha \in \mathbb{R}\,, \quad (p_i + p_j)^2 = (-1)^{i+j}\langle ij\rangle^2\,. \tag{2.38}$$

I adopt following light-cone combinations of spinor variables w_m^α, $v^{m\alpha}$ instead of usual particle momentum spinors λ_i^α

$$w_m^\alpha = \lambda_{2m}^\alpha + \lambda_{2m-1}^\alpha\,, \quad v^{m\alpha} = \lambda_{2m}^\alpha - \lambda_{2m-1}^\alpha\,. \tag{2.39}$$

The light-cone basis is much convenient for amplitudes computation. To avoid confusion, I reserve the notation $\langle 12\rangle = \langle \lambda_1 \lambda_2\rangle$ exclusively for the particle basis. In the light-cone basis, I will use $\langle w_m w_n\rangle$, $\langle w_m v^n\rangle$ and so on. The light-cone components of the fermionic coordinates η_i^I are denoted by

$$\bar{\theta}_m^I = \eta_{2m}^I + \eta_{2m-1}^I\,, \quad \theta^{mI} = \eta_{2m}^I - \eta_{2m-1}^I\,. \tag{2.40}$$

In the light-cone basis, the overall momentum conservation and the super-momentum components are written as

$$P^{\alpha\beta} = \sum_{i=1}^{2k}(-1)^i\lambda_i^\alpha\lambda_i^\beta = \frac{1}{2}\left(w_m^\alpha v^{m\beta} + w_m^\beta v^{m\alpha}\right) = 0\,, \tag{2.41}$$

$$Q^{\alpha I} = \sum_{i=1}^{2k}(-1)^i\lambda_i^\alpha\eta_i^I = \frac{1}{2}\left(w_m^\alpha\theta^{mI} + v^{m\alpha}\bar{\theta}_m^I\right)\,, \tag{2.42}$$

where I used summation convention for the light-cone coordinates. The metric and the kinetic variables in the light-cone basis take the following form

$$g = \frac{1}{2}\begin{pmatrix} 0 & \delta^m{}_n \\ \delta_m{}^n & 0 \end{pmatrix}\,, \quad \lambda = \begin{pmatrix} w_n \\ v^n \end{pmatrix}\,, \quad \eta = \begin{pmatrix} \bar{\theta}_n \\ \theta^n \end{pmatrix}\,. \tag{2.43}$$

U-gauge

The light-cone form of the $(k \times 2k)$ C-matrix before a gauge fixing is

$$C = \left(t_a{}^n \mid s_{an}\right)\,, \tag{2.44}$$

and using the $GL(k)$ redundancy one can obtain the following gauge fixed C-matrix

$$C = \left(t_a{}^n \mid s_{an}\right) \quad \rightarrow \quad \left(\delta_m{}^n \mid u_{mn}\right)\,, \quad u_{mn} = (t^{-1})_m{}^a s_{an}\,. \tag{2.45}$$

Under such gauge fixed form of C-matrix, the orthogonality condition reduce to anti-symmetric conditions for u_{mn}

$$\delta(C \cdot g \cdot C^T) = \delta\left(\frac{1}{2}(u_{mn} + u_{nm})\right)\,. \tag{2.46}$$

Since the decomposition of u_{mn} is a linear operation, there is no u_{mn} dependent non-trivial Jacobian factor. I call such gauge choice the "u-gauge". This u-gauged C-matrix was inspired by the work in ref. [69] which studied the orthogonal Grassmannian and the pure spinor. They admit the same $SO(2k)/U(k)$ coset description, and the pure spinors can be understood as higher-dimensional twistors and u_{mn} coordinates were used to solve the non-linear pure spinors constraints equation.

The u-gauge has many advantages. As I already mentioned above, it automatically solve the orthogonality condition by imposing the anti-symmetric condition for u_{mn}. Another thing is that with the u-gauge and the light-cone basis bosonic delta function $C \cdot \lambda = 0$ is written as

$$w_m + u_{mn} v^n = 0\,, \tag{2.47}$$

and this equation admits a particular form of solution for all k

$$u^*_{mn} = \frac{2\langle w_m w_n \rangle}{\langle w_p v^p \rangle} = \frac{1}{R}\langle w_m w_n \rangle\,, \quad R \equiv \frac{\langle w_p v^p \rangle}{2}\,. \tag{2.48}$$

One can verify (2.48) is indeed a solution to (2.47) by the Schouten identity

$$\langle w_m w_n \rangle v^{p\alpha} + \langle w_n v^p \rangle w^\alpha_m + \langle v^p w_m \rangle w^\alpha_n = 0\,, \tag{2.49}$$

and the momentum conservation (2.42).

Dual U-gauge

I shortly want to mention the "dual u-gauge". The u-gauge is not the only gauge choice, but it is natural to think the "dual u-gauge". In this gauge, the role of w_n and v^n are reversed

$$C = \left({x_a}^n \mid u_{an} \right) \quad \to \quad \left(\bar{u}^{mn} \mid {\delta^m}_n \right)\,. \tag{2.50}$$

In the dual u-gauge, the $C \cdot \lambda = 0$ condition reads

$$v^m + \bar{u}^{mn} w_n = 0\,, \tag{2.51}$$

which also admits a particular dual solution,

$$\bar{u}_*^{mn} = -\frac{1}{R}\langle v^m v^n \rangle\,. \tag{2.52}$$

The dual u-gauge will appear in λ-parity for odd k. Together with the original and dual u-gauge solutions, one can obtain identities that will be useful in amplitude computations

$$u^*_{mp}\bar{u}_*^{np} = -\frac{1}{R}\langle w_m v^n \rangle\,, \quad u^*_{mp}\bar{u}_*^{mp} = -2\,. \tag{2.53}$$

One can prove the identities using the energy momentum conservation and Schouten identity.

Broader class of u-gauges

As learned from the ordinary gauge choice of C-matrix, many properties of the u-gauge remain same under permutations of columns. For instance, instead of alternative signature metrics (2.37), one can choose

$$g = \mathrm{diag}(\underbrace{-,\cdots,-}_{k},\underbrace{+,\cdots,+}_{k})\,, \tag{2.54}$$

and define light-cone coordinates by

$$w_m = (-1)^{k-1}(x_{m+k} + x_m)\,, \quad v^m = (-1)^{k-1}(x_{m+k} - x_m)\,. \tag{2.55}$$

As one can see in above, u-gauge doesn't produce any explicit u_{mn} dependent Jacobian factor when solving the orthogonality conditions. They only produce

R-dependent Jacobians when solving the kinematic and fermionic delta function.

One can choose arbitrary the diagonal entries of the metric with k (-1) and k $(+1)$ entries. There are $C_k^{2k} = (2k)!/(k!)^2$ different such metrics. An overall sign flip is irrelevant, so there are $(2k-1)!/(k!(k-1)!)$ inequivalent metrics. If one chooses an metric, there are $k!$ inequivalent ways of choosing light-cone coordinate. As a result, there are $(2k-1)!/(k-1)!$ u-type gauges.

Among all possibilities, I am interested in called 'u-cyclic gauge' and 'u-factorization gauge'. The u-cyclic gauge choice is the case with the metric (2.37) and the light-cone basis (2.39). The u-factorization gauge choice is the case with the metric (2.54) and the light-cone basis (2.55).

General solutions

The number of integral variables in orthogonal Grassmannian of ABJM amplitude is reduced to $(k-2)(k-3)/2$ by kinematic delta functions. So for $k=2$ and $k=3$ cases, (2.48) is the unique solution to (2.47). For $k=4$, the general solution for the kinematic delta functions can be parametrized by

$$\hat{u}_{mn}(z) = u^*_{mn} + \frac{z}{2}\epsilon_{mnrs}\bar{u}^{rs}_* \,. \tag{2.56}$$

The general solution for $k=5$ is

$$\hat{u}_{mn}(z) = u^*_{mn} + \frac{z^p}{2}\epsilon_{mnprs}\bar{u}^{rs}_* \,. \tag{2.57}$$

At first glance, the "vector" z^p appears to have five components. But only three of them are independent because of the following equivalence relation,

$$z^p \sim z^p + c_\alpha v^{p\alpha} \,, \tag{2.58}$$

which follows from the fact that $\bar{u}_*^{rs} \propto \langle v^m v^n \rangle$ and the Schouten identity. At the level of actual computation, one can just set $z^4 = 0$ and $z^5 = 0$(or any other choices). This can be generalized into the following form of general solution for $k \geq 4$

$$\hat{u}_{mn}(z) = u^*_{mn} + \frac{1}{2(k-4)!} z^{p_1 \cdots p_{k-4}} \epsilon_{mnp_1 \cdots p_{k-4} rs} \bar{u}_*^{rs} \,. \tag{2.59}$$

Lambda-parity in the u-gauge

As I mentioned earlier, momentum $p^{\alpha\beta}$ is invariant under the little group $\mathbb{Z}_2$ action on the momentum spinor $\lambda^\alpha \to -\lambda^\alpha$, so it called λ-parity. This $\mathbb{Z}_2$ action flips the sign of the fermion wave-function. I will study how the lambda parity is reflected in the u-gauge and show that the lambda parity, for odd k, induces the exchange,

$$w_m \quad \leftrightarrow \quad v^m \,. \tag{2.60}$$

Below argument does not rely on the choice of gauge. So I will use the u-factorization gauge for simplicity. Let's first consider a usual $GL(k)$ gauge fixed C-matrix in the particle basis [6, 24],

$$C = \Big(I \mid O \Big) \,, \qquad OO^T = I \,. \tag{2.61}$$

In the light-cone basis, the C-matrix translates to

$$\hat{C} = \Big(I - O \mid I + O \Big) \,. \tag{2.62}$$

For odd-dimensional orthogonal matrices, one can use following identities

$$\det(O) = \pm 1 \quad \Longrightarrow \quad \det(I \mp O) = 0 \,. \tag{2.63}$$

One can prove the above identity by acting $1 = \pm\det(O)$ to the $\det(I \mp O)$. When $\det(O) = -1$, a $GL(k)$ gauge transformation gives

$$\hat{C} \to \Big(I \mid (I-O)^{-1}(I+O)\Big) = \Big(I \mid U\Big) \,. \tag{2.64}$$

This is the relation between the u-gauge and the usual gauge in the particle basis.

The other branch $\det(O) = +1$ is related to the $\det(O) = -1$ branch by $\det(-O) = -\det(O)$, and such sign flip has the same effect as flipping the signs of all λ_m^α for $m = k+1, \cdots, 2k$. Then known fact for OG_{2k} is that two different branches in OG_{2k} are related with λ-parity. Up to an overall $SO(2k)$ rotation, this is the same as the exchange (2.60). As a result, One can see the λ-parity induces the exchange of w_m and v^m for odd k. This result will be useful to the 6-point amplitude calculation.

2.3.2 4,6-point amplitude

In the previous subsection, I have discussed about the new choice of gauge called u-gauge. In this subsection, I will reproduce the 4-point and 6-point ABJM amplitudes using the u-gauge. The previous results are reproduced more simply and symmetrically through the u-gauge. One can also find physical pole structures which hard to see in old gauge choices for the 6-point amplitude.

4-point amplitude

The momentum conservation with split signature is given by

$$P^{\alpha\beta} = -\lambda_1^\alpha\lambda_1^\beta + \lambda_2^\alpha\lambda_2^\beta - \lambda_3^\alpha\lambda_3^\beta + \lambda_4^\alpha\lambda_4^\beta = 0\,. \tag{2.65}$$

Then, one can find following identities in 4-point kinematics

$$\frac{\langle 12\rangle}{\langle 34\rangle} = \frac{\langle 23\rangle}{\langle 14\rangle} = -\frac{\langle 31\rangle}{\langle 24\rangle} = \sigma = \pm 1\,, \tag{2.66}$$

$$\langle 12\rangle^2 + \langle 23\rangle^2 - \langle 31\rangle^2 = 0\,. \tag{2.67}$$

The sign factor σ in (2.66) specifies a branch of OG_4. One can take $\sigma = +1$ branch without loss of generality. In the light-cone basis, C-matrix with the u-gauge($u = u_{12}$) and the metric are given by

$$\widehat{C} = \begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & -u & 0 \end{pmatrix}\,, \quad \hat{g} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\,. \tag{2.68}$$

In the particle basis,

$$C = \begin{pmatrix} 1 & 1 & -u & u \\ u & -u & 1 & 1 \end{pmatrix}\,, \quad g = \mathrm{diag}(-,+,-,+)\,. \tag{2.69}$$

In this gauge, the orthogonality condition is automatically solved with trivial Jacobian factor, and the kinematic delta-functions reduce to the following

$$\delta(C\cdot\lambda) = J_4^B\,\delta^3(P)\,\delta(u-u_*)\,. \tag{2.70}$$

The solution of u_* is already given in (2.48)

$$u_* = \frac{2\langle w_1 w_2\rangle}{\langle w_1 v^1\rangle + \langle w_2 v^2\rangle} = \frac{\langle 23\rangle - \langle 31\rangle}{\langle 12\rangle} = -\frac{\langle 12\rangle}{\langle 23\rangle + \langle 31\rangle}\,. \tag{2.71}$$

I have used (2.66) with $\sigma = +1$ and (2.67). One can read the Jacobian factor in (2.70)

$$J_4^B = \frac{\langle v^1 v^2\rangle}{2} = -(\langle 23\rangle + \langle 31\rangle)\,. \tag{2.72}$$

The fermionic delta function is given as

$$\delta(C \cdot \eta) = J_4^F \delta^6(Q) = \left(\frac{4}{\langle v^1 v^2 \rangle} \right)^3 \delta^6(Q) \,. \tag{2.73}$$

The denominator at $u = u_*$ is

$$\left. \frac{1}{M_1 M_2} \right|_{u=u_*} = \frac{1}{(-2u_*)(1-u_*^2)} = \frac{(\langle 23 \rangle + \langle 31 \rangle)^2}{4\langle 12 \rangle \langle 23 \rangle} \,. \tag{2.74}$$

Finally, one can reproduce the standard 4-point amplitude after collecting all ingredients

$$\mathcal{A}_4 = \frac{\delta^3(P)\delta^6(Q)}{\langle 12 \rangle \langle 23 \rangle} \,. \tag{2.75}$$

6-point amplitude

The Grassmannian integral for the 6-point amplitude is also fully localized by the delta functions, i.e. there is no contour integral. In the particle basis with split signature, the gauge-fixed C matrix in the $(+)$-branch is

$$C^+ = \begin{pmatrix} 1 & 1 & -u_{12} & u_{12} & -u_{13} & u_{13} \\ u_{12} & -u_{12} & 1 & 1 & -u_{23} & u_{23} \\ u_{13} & -u_{13} & u_{23} & -u_{23} & 1 & 1 \end{pmatrix} \,. \tag{2.76}$$

The kinematic delta-function can be written as

$$\delta(C \cdot \lambda) = J_6^B \, \delta^3(P) \, \delta^3(u_{mn} - u_{mn}^*) \,. \tag{2.77}$$

The general solution of the bosonic delta function is given by (2.48) with the Jacobian factor $J_6^B = 1/2$. The fermionic delta function is reduced to the

$$\delta(C \cdot \eta) = J_6^F \delta^6(Q) \delta^3(\zeta_+) \,, \quad \zeta_+^I = \frac{1}{2} \epsilon^{mnp} u_{mn}^* \bar{\theta}_p^I \,, \tag{2.78}$$

where the Jacobian factor for the fermionic delta function is obtained by

$$J_6^F = \left(\frac{4}{R}\right)^3 , \tag{2.79}$$

with R defined in (2.48). The denominator at $u_{ij} = u_{ij}^*$ is

$$\frac{1}{M_1^+ M_2^+ M_3^+}\bigg|_{u_{mn}=u^*_{mn}} = \frac{1}{8(u^*_{13} - u^*_{12}u^*_{23})(u^*_{23} - u^*_{12}u^*_{13})(u^*_{12} - u^*_{13}u^*_{23})} . \tag{2.80}$$

Collecting all ingredients, the 6-point Grassmannian integral in the u-gauge becomes

$$\frac{4\,\delta^3(P)\delta^6(Q)\delta^3(\zeta_+)}{R^3(u^*_{13} - u^*_{12}u^*_{23})(u^*_{23} - u^*_{12}u^*_{13})(u^*_{12} - u^*_{13}u^*_{23})} . \tag{2.81}$$

The full amplitude is given by sum of the two branches of OG_6. Two branches are related by λ-parity. As I already explained, for odd k, λ-parity is nothing but the exchange of the two types of light-cone coordinates. In terms of the super-space variables, the exchange means

$$w_m \leftrightarrow v^m , \quad \bar{\theta}_m \leftrightarrow \theta^{mI} , \quad \zeta_+^I \leftrightarrow \zeta_-^I = \frac{1}{2}\epsilon_{mnp}\bar{u}_*^{mn}\theta^{pI} . \tag{2.82}$$

The explicit form of the C-matrix in the conjugate branch is given by

$$C^- = \begin{pmatrix} -1 & 1 & \bar{u}^{12} & \bar{u}^{12} & \bar{u}^{13} & \bar{u}^{13} \\ -\bar{u}^{12} & -\bar{u}^{12} & -1 & 1 & \bar{u}^{23} & \bar{u}^{23} \\ -\bar{u}^{13} & -\bar{u}^{13} & -\bar{u}^{23} & -\bar{u}^{23} & -1 & 1 \end{pmatrix} . \tag{2.83}$$

Summing up the two terms, I obtain the full 6-point amplitude

$$\begin{aligned}\mathcal{A}_6 =& \delta^3(P)\delta^6(Q)\frac{4}{R^3}\left(\frac{\delta^3(\zeta_+)}{(u^*_{13} - u^*_{12}u^*_{23})(u^*_{23} - u^*_{12}u^*_{13})(u^*_{12} - u^*_{13}u^*_{23})}\right. \\ &\left. + \frac{\delta^3(\zeta_-)}{(\bar{u}_*^{13} - \bar{u}_*^{12}\bar{u}_*^{23})(\bar{u}_*^{23} - \bar{u}_*^{12}\bar{u}_*^{13})(\bar{u}_*^{12} - \bar{u}_*^{13}\bar{u}_*^{23})}\right) ,\end{aligned} \tag{2.84}$$

The 6-point amplitude is known to have three different factorization channels. Under the factorization limit, the 6-point amplitude can be factorized into two 4-point amplitudes. The factorization property is not manifest in the (2.84). But I have found that product of the two minors in different two branches produces the physical poles, and they are given by

$$M_1^+ M_1^- = \left(\frac{4}{R}\right)^2 p_{123}^2 \,, \quad M_2^+ M_2^- = -\left(\frac{4}{R}\right)^2 p_{234}^2 \,, \quad M_3^+ M_3^- = \left(\frac{4}{R}\right)^2 p_{345}^2 \,, \tag{2.85}$$

where $p_{ijk}^2 = (p_i + p_j + p_k)^2$.

One can check, of course, the above equalities by numerics. This is the analytic proof for one of the above equalities

$$\begin{aligned}
4p_{123}^2 &= -\left(\langle 12\rangle^2 + \langle 23\rangle^2 - \langle 13\rangle^2\right) \\
&= -\frac{1}{4}\left(\langle w_1 - v^1, w_1 + v^1\rangle^2 + \langle w_1 + v^1, w_2 - v^2\rangle^2 - \langle w_1 - v^1, w_2 - v^2\rangle^2\right) \\
&= -\left(\langle w_1 v^1\rangle^2 - (\langle w_1 w_2\rangle - \langle w_1 v^2\rangle)(\langle w_2 v^1\rangle + \langle v^1 v^2\rangle)\right) \\
&= -R^2\left((u_{12}\bar{u}^{12} + u_{13}\bar{u}^{13})^2 + (u_{12} + u_{13}\bar{u}^{23})(\bar{u}^{12} + u_{23}\bar{u}^{13})\right) \\
&= -R^2\Big((u_{12}\bar{u}^{12})^2 + (u_{13}\bar{u}^{13})^2 + 2u_{12}\bar{u}^{12}u_{13}\bar{u}^{13} + u_{12}\bar{u}^{12} + u_{13}\bar{u}^{13}u_{23}\bar{u}^{23} \\
&\qquad\quad + u_{13}\bar{u}^{12}\bar{u}^{23} + \bar{u}^{13}u_{12}u_{23}\Big) \\
&= -R^2\Big(u_{12}\bar{u}^{12}(-1 - u_{13}\bar{u}^{13} - u_{23}\bar{u}^{23}) + u_{13}\bar{u}^{13}(-1 - u_{12}\bar{u}^{12} - u_{23}\bar{u}^{23}) \\
&\qquad + 2u_{12}\bar{u}^{12}u_{13}\bar{u}^{13} + u_{12}\bar{u}^{12} + u_{13}\bar{u}^{13}u_{23}\bar{u}^{23} + u_{13}\bar{u}^{12}\bar{u}^{23} + \bar{u}^{13}u_{12}u_{23}\Big) \\
&= -R^2\Big(- u_{13}\bar{u}^{13} - u_{12}\bar{u}^{12}u_{23}\bar{u}^{23} + u_{13}\bar{u}^{12}\bar{u}^{23} + \bar{u}^{13}u_{12}u_{23}\Big) \\
&= \frac{R^2}{4} M_1^+ \, M_1^- \,.
\end{aligned} \tag{2.86}$$

Here I used the definition of the light-cone basis spinors and the equalities given in (2.53). I find that above pole structures can be generalized with following relation for product of two minors

$$M^+_{ijk}M^-_{ijk} = \left(\frac{4}{R}\right)^2 (-1)^{i+j+k} p^2_{ijk}\,, \tag{2.87}$$

where the subscripts (ijk) denote a set of three columns from the C-matrix, and the superscript $(\pm)$ distinguishes the two branches.

2.3.3 8-point amplitude

The Grassmannian formula for the 8-point amplitude has one complex contour integral. I will show how to do this contour integral in the u-gauge condition.

Contour integral

As I mentioned earlier, the general solution to the 8-point $C \cdot \lambda = 0$ in the u-gauge is

$$\hat{u}_{mn}(z) = u^*_{mn} + \frac{z}{2}\epsilon_{mnpq}\bar{u}^{pq}_* \,. \tag{2.88}$$

The Grassmannian integral reduces to a contour integral in z

$$\delta^8(C\cdot\lambda) = J^B_8\; \delta^3(P) \int dz\; \delta^6(u_{mn} - \hat{u}_{mn}(z))\,, \tag{2.89}$$

with $J^B_8 = 1/(2R)$. Contour description was already discussed in [6], and I choose the contour that picks the poles from M_1 and M_3. One can choose another contour that picks the poles of M_2 and M_4, and it will give same result with the previous contour choice up to overall sign.

At first glance, the minors of C-matrix can be quartic in $\hat{u}_{mn}(z)$. But, explicit computations show that all quartic and cubic terms disappear by the following identity

$$\hat{u}_{mn}(z)\hat{u}_{pq}(z) + \hat{u}_{mp}(z)\hat{u}_{qn}(z) + \hat{u}_{mq}(z)\hat{u}_{np}(z) = -z\,\epsilon_{mnpq}\,. \tag{2.90}$$

Finally, one can see the all consecutive minors are given by quadratic equation of z

$$M_i(z) = a_i z^2 + b_i z + c_i = a_i(z - z_i^+)(z - z_i^-)\,, \tag{2.91}$$

where a_i, b_i, c_i are function of u^*_{ij} and $\bar{u}^{ij}_*$. One can also extract explicit super-momentum conservation factor from the fermionic delta function

$$\delta^{12}(C\cdot\eta) = \left(\frac{4}{R}\right)^3 \delta^6(Q) \prod_{I=1}^{3}(A_I z^2 + B_I z + C_I)\,, \tag{2.92}$$

where the fermion bilinears (A_I, B_I, C_I) are defined as follows

$$\begin{aligned}
&\frac{1}{4}\epsilon^{mnpq}u^*_{mn}\left(\bar{\theta}^I_p + \frac{1}{2}z\epsilon_{prxy}\bar{u}^{xy}_*\theta^{rI}\right)\left(\bar{\theta}^I_q + \frac{1}{2}z\epsilon_{qszw}\bar{u}^{zw}_*\theta^{sI}\right) \\
&= \frac{1}{4}\left[-z^2\epsilon_{mnpq}\bar{u}^{mn}_*\theta^{pI}\theta^{qI} + z\epsilon^{mnpq}\epsilon_{prxy}\bar{u}^*_{mn}\bar{u}^{xy}_*\theta^{rI}\bar{\theta}^I_q + \epsilon^{mnpq}u^*_{mn}\bar{\theta}^I_p\bar{\theta}^I_q\right] \\
&= A_I z^2 + B_I z + C_I\,.
\end{aligned} \tag{2.93}$$

The variables A_I, B_I, C_I are now considered as commuting bosonic variables. Now the problem reduce to evaluate the contour integral

$$I(\mathcal{C}) = \oint_{\mathcal{C}} \frac{dz}{2\pi i}\frac{\prod_{I=1}^{3}(A_I z^2 + B_I z + C_I)}{\prod_{i=1}^{4}(a_i z^2 + b_i z + c_i)}\,. \tag{2.94}$$

Note that, the distinction between (a_i, b_i, c_i) and (A_I, B_I, C_I) is not important at the level of computation, because computational details do not depend on (a_i, b_i, c_i) and (A_I, B_I, C_I). For convenience, I replace (A_I, B_I, C_I) $(I = 1, 2, 3)$ by (a_i, b_i, c_i) $(i = 5, 6, 7)$ in what follows. If one naively do the contour integral, many square roots will appear in the middle of computation. Such square roots must be canceled in the final result, because tree-level scattering amplitude is the analytic function of Lorentz invariants. However, it is difficult to see the cancellation due to the complexity of the integrand. One need to do the contour integral more clever way.

I consider following family of integrals to illustrate the ideas of my computation technique

$$I_n(\mathcal{C}) = \oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{\prod_{j=n+2}^{2n+1}(a_j z^2 + b_j z + c_j)}{\prod_{i=1}^{n+1}(a_i z^2 + b_i z + c_i)} \,. \tag{2.95}$$

The choice $n = 3$ recovers the original integral.

These integrals have following two interesting features. One is that they are homogeneous functions of the variables (a_i, b_i, c_i) with degree (-1) for $i = 1, \ldots, n+1$ and $(+1)$ for $i = n+2, \ldots, 2n+1$. The other is that they are invariant under the $SL(2, \mathbb{C})$ transformation,

$$z \to \frac{ez + f}{gz + h}\,, \qquad \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in SL(2, \mathbb{C})\,. \tag{2.96}$$

$SL(2, \mathbb{C})$ transformation has following generators:

$$\text{Dilatation:} \quad z \to e^t z\,, \tag{2.97}$$

$$\text{Inversion:} \quad z \to -1/z\,, \tag{2.98}$$

$$\text{Translation:} \quad z \to z + \epsilon\,. \tag{2.99}$$

The change in z can be reproduced by the following changes of the coefficients:

$$\text{Dilatation:} \quad (a_i, b_i, c_i) \to (e^t a_i, b_i, e^{-t} c_i)\,, \tag{2.100}$$

$$\text{Inversion:} \quad (a_i, b_i, c_i) \to (c_i, -b_i, a_i)\,, \tag{2.101}$$

$$\text{Translation:} \quad (a_i, b_i, c_i) \to (a_i, b_i + 2a_i\epsilon, c_i + b_i\epsilon + a_i\epsilon^2)\,. \tag{2.102}$$

My computation technique is based on the above transformation rule. The integral (2.95) should be invariant under the $SL(2, \mathbb{C})$ action on z. This requires that final result should be invariant under the $SL(2, \mathbb{C})$ action of the (a_i, b_i, c_i)

also. Finally, the problem can be translated into the inspection of the possible $SL(2,\mathbb{C})$ object which satisfy the all properties of the integral.

To be specific, let us focus on the contour $\mathcal{C}_1$ which encloses the two poles $z_1^\pm$ only. The residue theorem gives

$$\begin{aligned}I_n(\mathcal{C}_1) &= \frac{1}{a_1}\left(\frac{N_n(z_1^+)}{D_n(z_1^+)} - \frac{N_n(z_1^-)}{D_n(z_1^-)}\right)\frac{1}{z_1^+ - z_1^-} \\ &= \frac{1}{a_1 D_n(z_1^+)D_n(z_1^-)}\cdot\frac{N_n(z_1^+)D_n(z_1^-) - N_n(z_1^-)D_n(z_1^+)}{z_1^+ - z_1^-}\,,\end{aligned} \tag{2.103}$$

where I defined

$$N_n(z) = \prod_{j=n+2}^{2n+1} M_j(z)\,, \quad D_n(z) = \prod_{i=2}^{n+1} M_i(z)\,, \quad M_i(z) = a_i z^2 + b_i z + c_i\,. \tag{2.104}$$

The product $D_n(z_1^+)D_n(z_1^-)$ is easy to evaluate. Using the relations

$$z_1^+ + z_1^- = -\frac{b_1}{a_1}\,, \quad z_1^+ z_1^- = \frac{c_1}{a_1}\,, \tag{2.105}$$

I find

$$D_n(z_1^+)D_n(z_1^-) = a_1^{-2n}\prod_{i=2}^{n+1}\Delta_{1i}\,, \tag{2.106}$$

where I defined short-hand notations

$$\Delta_{ij} = \beta_{ij}^2 - \alpha_{ij}\gamma_{ij}\,, \quad \alpha_{ij} = b_i c_j - b_j c_i\,, \;\; \beta_{ij} = c_i a_j - c_j a_i\,, \;\; \gamma_{ij} = a_i b_j - a_j b_i\,. \tag{2.107}$$

The new symbols $(\alpha_{ij},\beta_{ij},\gamma_{ij})$ obey simple $SL(2,\mathbb{C})$ transformation rules,

$$\text{Dilatation:} \quad (\alpha_{ij},\beta_{ij},\gamma_{ij}) \to (e^{-t}\alpha_{ij},\beta_{ij},e^{t}\gamma_{ij})\,, \tag{2.108}$$

$$\text{Inversion:} \quad (\alpha_{ij},\beta_{ij},\gamma_{ij}) \to (\gamma_{ij},-\beta_{ij},\alpha_{ij})\,, \tag{2.109}$$

$$\text{Translation:} \quad (\alpha_{ij},\beta_{ij},\gamma_{ij}) \to (\alpha_{ij} - 2\beta_{ij}\epsilon + \gamma_{ij}\epsilon^2, \beta_{ij} - \gamma_{ij}\epsilon, \gamma_{ij})\,, \tag{2.110}$$

so that Δ_{ij} is fully invariant under $SL(2,\mathbb{C})$.

To summarize what I have done so far,

$$I_n(\mathcal{C}_1) = \frac{1}{\prod_{i=2}^{n+1}\Delta_{1i}} \cdot (a_1)^{2n-1} \frac{N_n(z_1^+)D_n(z_1^-) - N_n(z_1^-)D_n(z_1^+)}{z_1^+ - z_1^-}\,, \tag{2.111}$$

The remaining $z_1^\pm$-dependent part may look complicated as both $N_n(z)$ and $D_n(z)$ are degree $2n$ polynomials in z. But one can use the fact that their values are obtained at the two solutions of quadratic equation $M_1(z) = 0$. One can take the following polynomial quotients

$$N_n(z) = Q_n(z)M_1(z) + R_n z + S_n\,, \quad D_n(z) = \tilde{Q}_n(z)M_1(z) + \tilde{R}_n z + \tilde{S}_n\,. \tag{2.112}$$

Then the integral becomes

$$I_n(\mathcal{C}_1) = \frac{(a_1)^{2n-1}(R_n\tilde{S}_n - S_n\tilde{R}_n)}{\prod_{i=2}^{n+1}\Delta_{1i}}\,. \tag{2.113}$$

The denominator $\prod_{i=2}^{n+1}\Delta_{1i}$ has degree of $(2n; 2, \cdots, 2; 0, \cdots, 0)$. Now the problem reduces to find the numerator $F_n \equiv (a_1)^{2n-1}(R_n\tilde{S}_n - S_n\tilde{R}_n)$, which has degree $(2n-1; 1\cdots, 1; 1, \cdots, 1)$, in an $SL(2,\mathbb{C})$ invariant way. To do so, one needs to consider a few additional $SL(2,\mathbb{C})$-invariants:

$$\begin{aligned} J_{ijk} &= \det(a_i, b_j, c_k) = 6a_{[i}b_j c_{k]}\,, \\ K_{ij} &= b_i b_j - 2(a_i c_j + c_i a_j)\,, \\ L_{ijkl} &= \alpha_{ij}\gamma_{kl} - 2\beta_{ij}\beta_{kl} + \gamma_{ij}\alpha_{kl}\,. \end{aligned} \tag{2.114}$$

For $n = 0$, the integral vanishes trivially, because one can deformed the contour to infinity without encountering any other poles. For $n = 1$, the numerator F_n should be of degree $(1; 1; 1)$ in three groups of variables i.e. degrees of $(a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3)$ should be (1;1;1). F_n also anti-symmetric under the exchange of last two group of variables because of the origin of (2.111). J_{123} is the only $SL(2,\mathbb{C})$ invariant object which satisfies such properties. An explicit computation indeed shows that

$$F_1 = -J_{123}\,. \tag{2.115}$$

For $n = 2$, one needs to find a polynomial of degree $(3;1,1;1,1)$. Such F_2 should be totally symmetric under permutations in the same group because of the symmetry that integrand have, and it should be anti-symmetric under the exchange of the last two groups for the same reason with F_1 case. The object that satisfies such properties and be made by $SL(2,\mathbb{C})$ invariants is

$$F_2 = \frac{1}{2}K_{1(2}J_{3)1(4}K_{5)1} + \frac{1}{2}K_{11}J_{1(2}{}^{(4}K_{3)}{}^{5)}\,. \tag{2.116}$$

Finally, one needs to find similar object in $n = 3$ case, which is the original problem. An object which satisfies such properties is not unique. For these reason, one needs to inspect all such objects and compare all these with brute force contour integral result via *Mathetmatica*. The answer is

$$F_3 = -K_{11}J_{12}{}^{(5}J_{13}{}^{6}J_{14}{}^{7)} + \frac{3}{4}L^1{}_{(2}{}^{1}{}_{3}J_{4)1(5}L^1{}_{6}{}^{1}{}_{7)}\,. \tag{2.117}$$

8-point amplitude: the result

The 8-point amplitude can be written as

$$\mathcal{A}_8 = \delta^3(P)\delta^6(Q)(1+\pi)J_8^B J_8^F\left(\frac{F(1)}{\Delta_{12}\Delta_{13}\Delta_{14}} + \frac{F(3)}{\Delta_{31}\Delta_{32}\Delta_{34}}\right)\,, \tag{2.118}$$

with the bosonic and fermionic Jacobian factors

$$J_8^B = \frac{1}{2R}\,,\quad J_8^F = \left(\frac{4}{R}\right)^3\,. \tag{2.119}$$

As I mentioned earlier, Grassmannian formula only pick a specific branch from the two branches, so one needs to sum two branches of the orthogonal Grassmannian. With λ-parity operator π, the $(1+\pi)$ factor gives the sum of two branches. In (2.118), $F(i)$ denote the value of F_n in the contour $\mathcal{C}_i$

Physical and spurious poles

On-shell diagram pioneered by [12] gives the nice interpretation of the Grassmannian formulation as multi-BCFW gluing of 3-point amplitudes. Therefore, the on-shell diagram can be useful to see the pole structure of amplitudes. On-shell diagram for ABJM theory was studied in [28–30]. The 8-point amplitude (2.118) is given by the sum of two contour integrals, and its on-shell diagram representation is given in Figure 2.2,

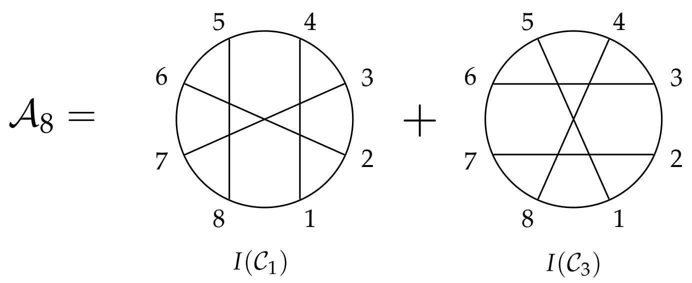


Figure 2.2 On-shell digrams for the 8-point amplitude collect residues from M_1 and M_3.

The on-shell diagram of ABJM amplitudes consist of two building blocks: a quartic vertex and an internal line. The quartic vertex is the smallest amplitude that ABJM theory can have, and it is given by the 4-point amplitude (2.75). The internal line which connects two diagrams equates the kinetic variables of two legs from different vertices and on-shell unlike Feynman diagram. On each internal line one should assign the integral rule $\int d^{2|3}\Lambda_I$.

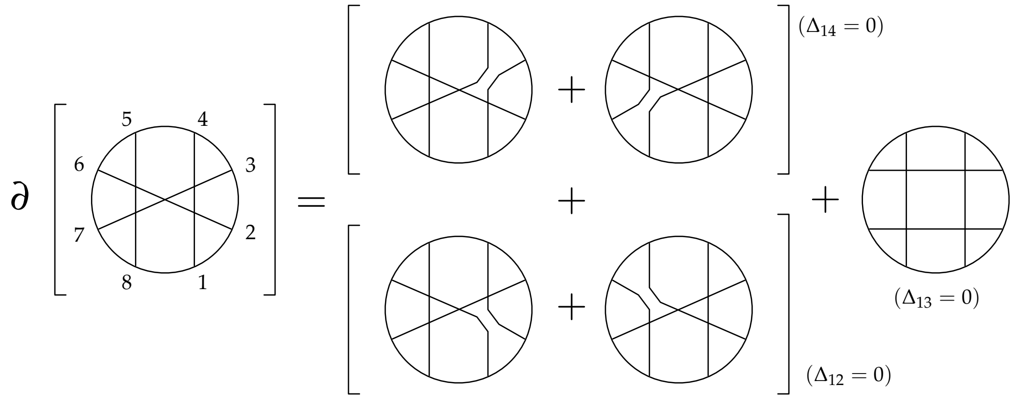


Figure 2.3 Poles of $I(C_1)$ correspond to boundary components of the on-shell diagram.

One can read the poles(physical or un-physical) of the amplitude from the boundaries of the on-shell diagrams. This boundary is given by the disconnecting the vertex, and the boundary operation of the on-shell diagram is drawn in Figure 2.3. It shows the five boundary terms from the on-shell diagram for $I(\mathcal{C}_1)$. One can see the physical or spurious(un-physical) poles. Now I will show how such poles(or boundaries) are related with vanishing condition of some minors. This procedure is highly non-trivial, and one can find the details in [29]. I just shortly introduce how to study the boundary of on-shell diagram. I adopt the coordinates of [29] associated with the OG tableaux

The tableau for $I(\mathcal{C}_1)$ is depicted in Figure 2.4. It can be translated to the C-matrix according to the rules explained in [29]. Let size k vector C_i be the i-th column of the C-matrix. Its components can be read from OG tableauxas by the following rules. I begin by setting the 'source' columns (C_1, C_2, C_3, C_5), and 'sink' columns (C_4, C_6, C_7, C_8), and assign a coordinate t_v to each vertex. I only consider the paths from a source to a sink that move upward and to the

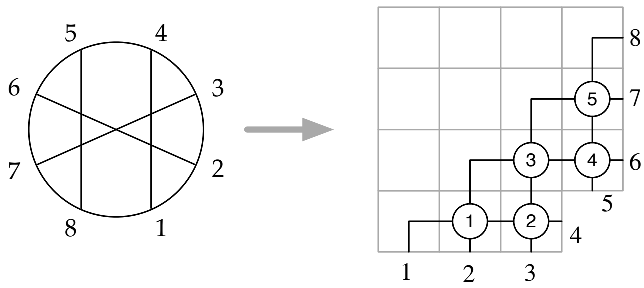


Figure 2.4 OG tableaux for $I(\mathcal{C}_1)$.

right but not downward or to the left. The path picks up $\pm\sinh(t_v)$ if it passes through the vertex, or $\pm\cosh(t_v)$ if it makes a turn at the vertex. The final matrix element is given by a polynomial of the form, schematically,

$$C_{\text{source,sink}} = \sum_{\text{paths}} \prod_{\text{vertices}} f_v \,. \tag{2.120}$$

The element of source to source is given by identity matrix $C_{i,j} = \delta_{ij}$. Using the above rule, one can obtain C-matrix, and it gives following consecutive minors

$$M_1 = 0\,, \quad M_2 = s_1 s_2\,, \quad M_3 = s_2 s_3 s_4\,, \quad M_4 = s_4 s_5\,, \quad s_v \equiv \sinh(t_v)\,. \tag{2.121}$$

The boundary operation corresponds with the coordinate variables becoming zero or infinity, and the untied diagram in the OG tableaux is shown in Figure 2.5.

One can rescale the minors by an overall factor, $\widetilde{M_i} = M_i/(s_2 s_4)$ to study the various factorization limit.

$$\widetilde{M_1} = 0\,, \quad \widetilde{M_2} = \frac{s_1}{s_4}\,, \quad \widetilde{M_3} = s_3\,, \quad \widetilde{M_4} = \frac{s_5}{s_2}\,. \tag{2.122}$$

The vanishing condition for $\widetilde{M_4}$ is translated into the vanishing limit of $s_2 \to \infty$ or $s_5 \to 0$. Such boundary limit is drawn in Figure 2.3, and channels for p_{123}^2

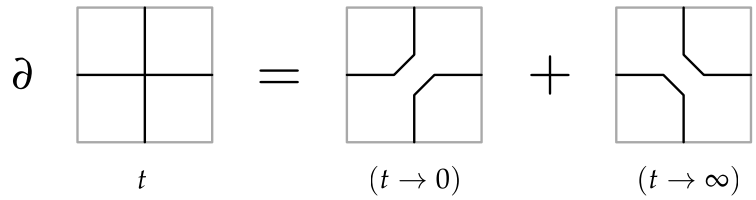


Figure 2.5 Boundary operation in terms of canonical coordinates.

and p_{567}^2 arise in this boundary. At the level of contour integral obtained earlier, the simultaneous vanishing limit of M_1 and M_4 is equivalent with the vanishing of limit Δ_{14}. As a result, it is natural to expect that Δ_{14} is proportional to $p_{123}^2 p_{567}^2$. In the u-gauge, one can explicitly verify the proportionality between Δ_{ij} and physical poles.

$$p_{123}^2 p_{567}^2 \propto R^4 \Delta_{14}\,, \;\; p_{234}^2 p_{678}^2 \propto R^4 \Delta_{12}\,, \;\; p_{345}^2 p_{781}^2 \propto R^4 \Delta_{32}\,, \;\; p_{456}^2 p_{812}^2 \propto R^4 \Delta_{34}\,. \tag{2.123}$$

The proof of (2.123) is given in Appendix A. One can identify the poles for $\Delta_{13} = \Delta_{31}$ in (2.118) as spurious poles. A standard argument in the Grassmannian integral uses the fact that

$$\mathcal{A}_8 = I(\mathcal{C}_1) + I(\mathcal{C}_3) = -I(\mathcal{C}_2) - I(\mathcal{C}_4)\,.$$

One can also think about spurious poles. Since $\Delta_{13} = \Delta_{31}$ arises from $I(\mathcal{C}_1)$ and $I(\mathcal{C}_3)$ but not from $I(\mathcal{C}_2)$ or $I(\mathcal{C}_4)$, it must be spurious. At the level of on-shell diagram, $\Delta_{13} = 0$ arises in the boundary of both $I(\mathcal{C}_1)$ and $I(\mathcal{C}_3)$, and they will be canceled if sign factors are properly assigned.

2.4 ABJM soft theorem

In this section, I will study the soft theorem for ABJM tree amplitudes. Soft theorems are to study the vanishing momenta limit of external photons(or gravitons). Such photons(or gravitons) which have zero-momenta are called soft photons(or gravitons), and they are undetectable in the physical experiment. The soft limit of tree amplitudes shows IR divergent of the theory, and it must be included in the cross-section computation to canceled the IR divergence arisen in the loop computation.

The IR divergent part in the soft theorem have a universal form. For this reason, it is sometimes used to confirm amplitude computations. It is the one of motivations for studying ABJM soft theorem. More recently, the sub-leading and sub-sub-leading soft graviton theorem and sub-leading photon theorem was studied via BCFW techniques [17,18], and it gives renewed interest in soft theorems of various theories and their applications.

2.4.1 ABJM double soft theorem

I use a similar analysis for three-dimensional supergravity theories [20] to derive ABJM soft theorem. First of all, it is natural to study the double soft limit of the $(2k+2)$-point amplitude $\mathcal{A}_{2k+2}$, since only even-point scattering amplitudes are well-defined as explained before. For simplicity, I take soft limit of last two external particles with small parameter ϵ

$$(p_{2k+1}, p_{2k+2}) \rightarrow \epsilon^2 (p_{2k+1}, p_{2k+2}) \,, \tag{2.124}$$

and taking the $\epsilon \to 0$ limit. In spinor variables, the scaling rule is

$$(\lambda_{2k+1}, \lambda_{2k+2}) \to \epsilon(\lambda_{2k+1}, \lambda_{2k+2})\,. \tag{2.125}$$

Under the above soft limit, $\mathcal{A}_{2k+2}$ reduces to the $\mathcal{A}_{2k}$ with a universal soft factor $S(\epsilon)$,

$$\mathcal{A}_{2k+2}|_{\epsilon\to 0} = S(\epsilon)\mathcal{A}_{2k}\,. \tag{2.126}$$

Finally, I will derive the leading and the sub-leading double soft theorem

$$\mathcal{A}_{2k+2}|_{\epsilon\to 0} = \left(\frac{1}{\epsilon^2}S^{(0)} + \frac{1}{\epsilon}S^{(1)}\right)\mathcal{A}_{2k} + \mathcal{O}(1)\,. \tag{2.127}$$

where the leading and sub-leading soft factors are

$$S^{(0)} = \frac{1}{2\langle 1, 2k\rangle}\left[\frac{\delta^3(\bar{\theta}_{k+1})}{\alpha_+\beta_+} - \frac{\delta^3(\theta_{k+1})}{\alpha_-\beta_-}\right], \tag{2.128}$$

$$\begin{aligned} S^{(1)} = {} & \frac{1}{2\langle 1, 2k\rangle\alpha_+\beta_+}\left[\frac{1}{2}\epsilon_{IJK}\bar{\theta}^I_{k+1}\bar{\theta}^J_{k+1}\xi^K_+ + \delta^3(\bar{\theta}_{k+1})\left(\beta_+ R_{2k+2,1} - \alpha_+ R_{2k+1,2k}\right)\right] \\ & + \frac{1}{2\langle 1, 2k\rangle\alpha_-\beta_-}\left[\frac{1}{2}\epsilon_{IJK}\theta^I_{k+1}\theta^J_{k+1}\xi^K_- + \delta^3(\theta_{k+1})\left(\beta_- R_{2k+2,1} + \alpha_- R_{2k+1,2k}\right)\right]. \end{aligned} \tag{2.129}$$

Various variables $\alpha_\pm, \beta_\pm, \chi_\pm, R_{i,j}$ will be explained later. Derivation of the soft theorem relies on the 3d on-shell recursion relation derived in Section 2.2.

2.4.2 Proof

Following the approach of ref. [20], I will use the BCFW recursion relation for ABJM amplitudes to obtain the double soft theorem. For convenience, I choose two shifted-particles in the BCFW recursion to be $(2k)$ and $(2k+1)$. As I explained in Section 2.2, the BCFW-shifted momentum spinors and on-shell superspace variables are given by

$$\hat{\lambda}_{2k} = c\lambda_{2k} + s\lambda_{2k+1}\,, \qquad \hat{\eta}_{2k} = c\eta_{2k} + s\eta_{2k+1}\,,$$
$$\hat{\lambda}_{2k+1} = s\lambda_{2k} + c\lambda_{2k+1}\,, \quad \hat{\eta}_{2k+1} = s\eta_{2k} + c\eta_{2k+1}\,. \tag{2.130}$$

where $c = \cosh t$ and $s = \sinh t$ with $c^2 - s^2 = 1$ and $z \equiv c + s = e^t$.[2]

One need to find IR divergent BCFW diagrams, and as explained in [17,20] for soft graviton theorems, only one factorization diagram contributes to the divergence. In my notation, the diagram is depicted in Figure 2.6. Let us briefly review why this is the case. The recursion formula was derived in Section 2.2, and they contain sum of many factorization diagrams and schematically takes the form

$$\mathcal{A}_{2k+2} = \sum_{l=2}^{k} \hat{\mathcal{A}}^{(L)}_{2l} \frac{H(z_+, z_-)}{p_f^2} \hat{\mathcal{A}}^{(R)}_{2(k+2-l)} + (z_+ \leftrightarrow z_-)\ . \tag{2.131}$$

When $l = 2$ as in Figure 2.6, in the soft limit $(p_{2k+1}, p_{2k+2}) \to \epsilon^2 (p_{2k+1}, p_{2k+2})$, p_f in Figure 2.6 becomes nearly equal to p_1 such that $1/p_f^2 \approx 1/p_1^2$ diverges by the on-shell condition of p_1. For $l > 2$ with generic external momenta, $1/p_f^2$ remains finite. The other factors $\hat{\mathcal{A}}^{(L)}_{2l}$, $\hat{\mathcal{A}}^{(R)}_{2(k+2-l)}$, $H(z_+, z_-)$ also remain finite.

Finally, One can figure out only in Figure 2.6 contribute to the soft theorem. Then problem becomes to evaluate the following amplitude

$$\begin{aligned}\mathcal{A}_{2k+2}|_{\epsilon\to 0} = \int d^3\eta_f \left[\mathcal{A}_4(\hat{\Lambda}_{2k+1}, \Lambda_{2k+2}, \Lambda_1, \hat{\Lambda}_f)(z_+) \frac{H(z_+, z_-)}{p^2_{1,2k+1,2k+2}} \mathcal{A}_{2k}(i\hat{\Lambda}_f, \cdots, \hat{\Lambda}_{2k})(z_+) \right] \\ + (z_+ \leftrightarrow z_-) \\ \equiv \mathcal{A}^{(+)}_{2k+2} + \mathcal{A}^{(-)}_{2k+2}\,. \end{aligned} \tag{2.132}$$

[2] The BCFW shift is slightly different in 2.2, because I am working on the split signature $g = \mathrm{diag}(-1, 1, \ldots, -1, 1)$

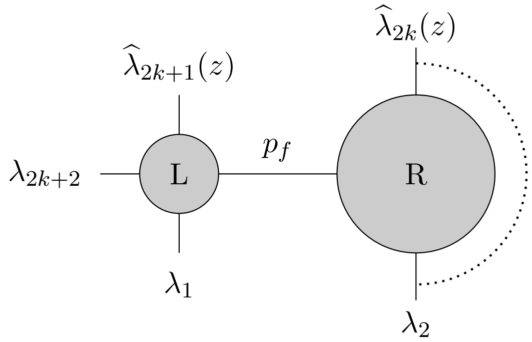


Figure 2.6 Soft limit factorization diagram.

In this diagram, the BCFW-deformed internal momentum p_f is given by

$$\hat{p}_f(z) = p_2 + p_3 + \cdots + p_{2k-1} + \hat{p}_{2k}(z) = -p_1 - p_{2k+2} - \hat{p}_{2k+1}(z)\,. \qquad (2.133)$$

The values z_+, z_- are the solutions of the on-shell condition of internal propagator, and general form of the solution was given in (2.28). In the soft limit, one can solve this condition order by order in ϵ by assuming that $s = s_0 + \epsilon s_1 + \epsilon^2 s_2 + \cdots$ and $c = c_0 + \epsilon c_1 + \epsilon^2 c_2 + \cdots$, with $c^2 - s^2 = 1$. Then the solutions are given by

$$c_\pm = 1 - \frac{\epsilon^2}{2}\alpha_\pm^2 + \mathcal{O}(\epsilon^4)\,, \qquad (2.134)$$

$$s_\pm = -\epsilon\alpha_\pm + \frac{\epsilon^3}{4}\left[(\alpha_\pm + \alpha_\mp)\alpha_\pm^2 - (\alpha_\pm - \alpha_\mp)\beta_\pm^2\right] + \mathcal{O}(\epsilon^5)\,, \qquad (2.135)$$

$$z_\pm = 1 - \epsilon\alpha_\pm - \frac{\epsilon^2}{2}\alpha_\pm^2 + \mathcal{O}(\epsilon^3)\,, \qquad (2.136)$$

where α_j and β_j are defined by

$$\alpha_\pm = \frac{\langle 1, 2k+1\rangle \pm \langle 1, 2k+2\rangle}{\langle 1, 2k\rangle}\,, \quad \beta_\pm = \frac{\langle 2k, 2k+1\rangle \pm \langle 2k, 2k+2\rangle}{\langle 1, 2k\rangle}\,. \qquad (2.137)$$

As mentioned in [20], the (+) solution corresponds to $\sigma = +1$, i.e.

$$\langle 1, f\rangle = \langle\widehat{2k+1}, 2k+2\rangle, \quad \langle f, \widehat{2k+1}\rangle = \langle 1, 2k+2\rangle, \quad \langle f, 2k+2\rangle = -\langle\widehat{2k+1}, 1\rangle, \qquad (2.138)$$

and the $(-)$ solution corresponds to $\sigma = -1$, i.e.,

$$\langle 1, f\rangle = -\langle \widehat{2k+1}, 2k+2\rangle, \ \ \langle f, \widehat{2k+1}\rangle = -\langle 1, 2k+2\rangle, \ \ \langle f, 2k+2\rangle = \langle \widehat{2k+1}, 1\rangle. \tag{2.139}$$

Note that, in (2.130), the first correction terms carry ϵ^2 weight relative to the leading terms for bosonic variables, whereas the relative weight is ϵ^1 for fermionic variables.[3] I want to derive not only leading soft factor but also sub-leading soft factor. To do this, one need to keep leading correction terms for momentum spinors, but I should keep leading and next to leading soft corrections for superspace variables.

Let's first focus on the $\mathcal{A}^{(+)}_{2k+2}$ part of (2.132). I will do the ϵ expansion for all terms in BCFW recursion formula. The internal propagator becomes

$$\frac{1}{(p_1 + p_{2k+1} + p_{2k+2})^2} = -\frac{1}{\epsilon^2}\frac{1}{\alpha_+\alpha_-\langle 1, 2k\rangle^2}\,. \tag{2.140}$$

The BCFW kernel gives

$$H(z_+, z_-) = \frac{z_+^2(z_-^2 - 1)}{z_+^2 - z_-^2} = \frac{\alpha_-}{\alpha_+ - \alpha_-}\,. \tag{2.141}$$

The bosonic factors that appear in the 4-point amplitude reduce to

$$\begin{aligned}
\langle f, \widehat{2k+1}\rangle &= \epsilon\langle 1, 2k+2\rangle\,,\\
\langle f, 2k+2\rangle &= -\langle \widehat{2k+1}, 1\rangle = -\epsilon\langle 1, 2k+2\rangle\,,\\
\langle f, 1\rangle &= \langle 2k+2, \widehat{2k+1}\rangle = \epsilon^2\langle 1, 2k+2\rangle\beta_+\,.
\end{aligned} \tag{2.142}$$

[3] λ_{2k+1} is rescaled $\epsilon\lambda_{2k+1}$, but η_{2k+1} does not change.

The fermionic delta function in the 4-point amplitude can be rewritten as

$$\begin{aligned}\delta^6(Q) &= \delta^3\left(\eta_f + \frac{\langle\widehat{2k+1}, 2k+2\rangle}{\langle\widehat{2k+1}, f\rangle}\eta_{2k+2} - \frac{\langle\widehat{2k+1}, 1\rangle}{\langle\widehat{2k+1}, f\rangle}\eta_1\right)\\ &\quad\times \delta^3(-\langle f, \widehat{2k+1}\rangle\hat{\eta}_{2k+1} + \langle f, 2k+2\rangle\eta_{2k+2} - \langle f, 1\rangle\eta_1)\\ &= \delta^3(\eta_f + \eta_1 + \epsilon\beta_+\eta_{2k+2})\\ &\quad\times (-\epsilon^3\langle 1, 2k+2\rangle^3) \times \delta^3(\eta_{2k+1} + \eta_{2k+2} + \epsilon(-\alpha_+\eta_{2k} + \beta_+\eta_1))\,.\end{aligned} \tag{2.143}$$

The first factor absorbs the fermionic integral in (2.132), and this contribution reproduces correct $2k$ reduced amplitude with ϵ correction. After combining all bosonic and fermionic factors in 4-point amplitude, it gives $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ contribution.

Collecting all ingredients, one can find the $\mathcal{A}^{(+)}_{2k+2}$ contribution to the soft limit

$$\mathcal{A}^{(+)}_{2k+2} = \frac{\delta^3(\tilde{\eta}_{2k+1} + \tilde{\eta}_{2k+2})}{2\epsilon^2\langle 1, 2k\rangle\alpha_+\beta_+}\mathcal{A}_{2k}(\tilde{\eta}_1, \eta_2, \eta_3, \cdots, \eta_{2k-1}, \tilde{\eta}_{2k})\,, \tag{2.144}$$

where

$$\begin{pmatrix}\tilde{\eta}_1\\ \tilde{\eta}_{2k+2}\end{pmatrix} = \begin{pmatrix}1 & \epsilon\beta_+\\ \epsilon\beta_+ & 1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_{2k+2}\end{pmatrix}, \quad \begin{pmatrix}\tilde{\eta}_{2k}\\ \tilde{\eta}_{2k+1}\end{pmatrix} = \begin{pmatrix}1 & -\epsilon\alpha_+\\ -\epsilon\alpha_+ & 1\end{pmatrix}\begin{pmatrix}\eta_{2k}\\ \eta_{2k+1}\end{pmatrix}. \tag{2.145}$$

Expanding explicitly in powers of ϵ, I obtain

$$\begin{aligned}\mathcal{A}^{(+)}_{2k+2} &= \frac{\delta^3(\bar{\theta}_{k+1})}{2\epsilon^2\langle 1, 2k\rangle\alpha_+\beta_+}\mathcal{A}_{2k}\\ &+\frac{1}{2\epsilon\langle 1, 2k\rangle\alpha_+\beta_+}\left[\frac{1}{2}\epsilon_{IJK}\bar{\theta}^I_{k+1}\bar{\theta}^J_{k+1}\xi^K_+ + \delta^3(\bar{\theta}_{k+1})\,(\beta_+R_{2k+2,1} - \alpha_+R_{2k+1,2k})\right]\mathcal{A}_{2k}\,.\end{aligned} \tag{2.146}$$

where I introduced

$$\xi_+ = -\alpha_+ \eta_{2k} + \beta_+ \eta_1 \,, \quad R_{i,j} = \eta_i \frac{\partial}{\partial \eta_j} \,. \tag{2.147}$$

The same result can be reproduce by choosing $(2k+2)$ and (1) as the reference legs for the BCFW recursion. In this sense, the symmetry between $(2k, 2k+1)$ and $(2k+2, 1)$ has been restored. The on-shell diagram perspective for such soft limit is illustrated in Figure 2.7.

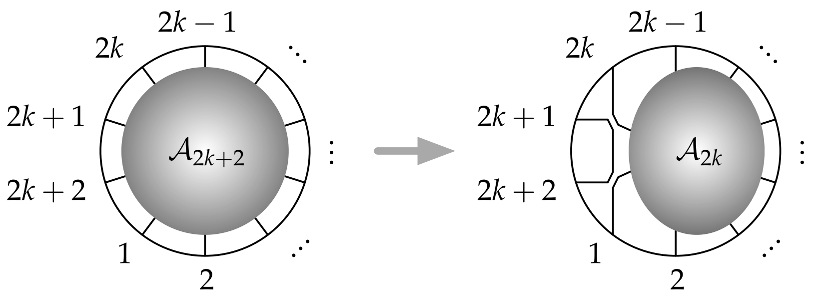


Figure 2.7 Soft limit from the on-shell diagram perspective.

The computation of $\mathcal{A}^{(-)}_{2k+2}$ in (2.132) proceeds in the same way. The result is

$$\begin{aligned}\mathcal{A}^{(-)}_{2k+2} = & -\frac{\delta^3(\theta_{k+1})}{2\epsilon^2\langle 1, 2k\rangle \alpha_- \beta_-} \mathcal{A}_{2k} \\ & + \frac{1}{2\epsilon\langle 1, 2k\rangle \alpha_- \beta_-} \left[\frac{1}{2}\epsilon_{IJK}\theta^I_{k+1}\theta^J_{k+1}\xi^K_- + \delta^3(\theta_{k+1})\left(\beta_- R_{2k+2,1} + \alpha_- R_{2k+1,2k}\right)\right] \mathcal{A}_{2k} \,.\end{aligned} \tag{2.148}$$

with $\xi_- = -\alpha_- \eta_{2k} + \beta_- \eta_1$. Note that $\mathcal{A}^{(+)}_{2k+2}$ and $\mathcal{A}^{(-)}_{2k+2}$ are related to each other by lambda-parity, $\Lambda_{2k+2} \to -\Lambda_{2k+2}$.

Finally, I obtain the leading and the sub-leading soft theorem for ABJM theory by adding two contributions $\mathcal{A}^{(+)}_{2k+2}$ and $\mathcal{A}^{(-)}_{2k+2}$

$$\mathcal{A}_{2k+2}|_{\epsilon\to 0} = \left(\frac{1}{\epsilon^2}S^{(0)} + \frac{1}{\epsilon}S^{(1)}\right)\mathcal{A}_{2k} + \mathcal{O}(1)\,. \tag{2.149}$$

where the leading and sub-leading soft factors are

$$S^{(0)} = \frac{1}{2\langle 1, 2k\rangle}\left[\frac{\delta^3(\bar\theta_{k+1})}{\alpha_+\beta_+} - \frac{\delta^3(\theta_{k+1})}{\alpha_-\beta_-}\right]\,, \tag{2.150}$$

$$\begin{aligned} S^{(1)} &= \frac{1}{2\langle 1, 2k\rangle\alpha_+\beta_+}\left[\frac{1}{2}\epsilon_{IJK}\bar\theta^I_{k+1}\bar\theta^J_{k+1}\xi^K_+ + \delta^3(\bar\theta_{k+1})\left(\beta_+R_{2k+2,1} - \alpha_+R_{2k+1,2k}\right)\right] \\ &\quad + \frac{1}{2\langle 1, 2k\rangle\alpha_-\beta_-}\left[\frac{1}{2}\epsilon_{IJK}\theta^I_{k+1}\theta^J_{k+1}\xi^K_- + \delta^3(\theta_{k+1})\left(\beta_-R_{2k+2,1} + \alpha_-R_{2k+1,2k}\right)\right]\,. \end{aligned} \tag{2.151}$$

2.4.3 Soft limit examples

I have derived double soft theorem for ABJM amplitudes. In this subsection, I will test the 6-point and 8-point amplitudes via the double soft theorem. As I mentioned, soft theorems can be used to test scattering amplitude results.

6-point amplitude soft limit

For simplicity, I use λ-parity operator π. For 6-point amplitude, it is given by

$$\pi\ :\ \Lambda_6 \to -\Lambda_6\,. \tag{2.152}$$

I consider only one part of the 6-point amplitude, and the other part will be obtained by the λ-parity operator. In the light-cone basis, soft limit with particle 5 and 6 soft is realized as

$$w_3 \to \epsilon\, w_3\,, \quad v^3 \to \epsilon\, v^3\,. \tag{2.153}$$

As one observed earlier, the leading correction of the bosonic kinematic invariants is ϵ^2 order. Therefore, it is natural to use the 4-point kinematic relations for particle (1,2,3,4) in the above soft limit. For instance, 6-point kinematic invariant R or identity for $u^*_{ij}, \bar{u}^{ij}_*$ are reduced to the 4-point one

$$R = \frac{1}{2}(\langle w_1 v^1\rangle + \langle w_2 v^2\rangle + \langle w_3 v^3\rangle) = \langle 12\rangle + \langle 34\rangle + \mathcal{O}(\epsilon^2) = 2\langle 12\rangle + \mathcal{O}(\epsilon^2)\,, \tag{2.154}$$

$$-1 = u^*_{12}\bar{u}^{12}_* + u^*_{13}\bar{u}^{13}_* + u^*_{23}\bar{u}^{23}_* = u^*_{12}\bar{u}^{12}_* + \mathcal{O}(\epsilon^2)\,. \tag{2.155}$$

In the soft limit, up to $\mathcal{O}(\epsilon^2)$ terms, the minors become

$$\begin{aligned} M_1^+ &= 2(u^*_{13} - u^*_{12}u^*_{23}) = 2u^*_{12}(-\bar{u}^{12}_* u^*_{13} - u^*_{23}) = \frac{2u^*_{12}}{R}(\langle w_3 v^2\rangle + \langle w_3 w_2\rangle) \\ &= \frac{2u^*_{12}}{R}(-2\epsilon)(\langle 45\rangle + \langle 46\rangle) = \epsilon\left(-\frac{4\langle 14\rangle u^*_{12}}{R}\right)\beta_+\,, \end{aligned} \tag{2.156}$$

$$\begin{aligned} M_2^+ &= -2(u^*_{23} - u^*_{12}u^*_{13}) = 2u^*_{12}(\bar{u}^{12}_* u^*_{23} + u^*_{13}) = \frac{2u^*_{12}}{R}(-\langle v^1 w_3\rangle + \langle w_1 w_3\rangle) \\ &= \frac{2u^*_{12}}{R}(2\epsilon)(\langle 15\rangle + \langle 16\rangle) = \epsilon\left(\frac{4\langle 14\rangle u^*_{12}}{R}\right)\alpha_+\,, \end{aligned} \tag{2.157}$$

$$M_3^+ = -2(u^*_{12} - u^*_{13}u^*_{23}) = -2u^*_{12}\,. \tag{2.158}$$

The 6-point amplitude (2.84) contains two fermionic parts

$$\delta^3(\zeta_+)\,, \quad \delta^6(Q_6)\,. \tag{2.159}$$

Neglecting $\mathcal{O}(\epsilon^2)$ terms, one can observe that

$$\delta^3(\zeta_+) = (u^*_{12})^3\delta^3\left(\bar{\theta}_3 + \frac{u^*_{23}}{u^*_{12}}\bar{\theta}_1 - \frac{u^*_{13}}{u^*_{12}}\bar{\theta}_2\right) = (u^*_{12})^3\delta^3(\bar{\theta}_3 + \epsilon(-\alpha_+\eta_4 + \beta_+\eta_1))\,. \tag{2.160}$$

The second identity follows from

$$\begin{aligned}
\frac{u_{23}^*}{u_{12}^*}\bar{\theta}_1 - \frac{u_{13}^*}{u_{12}^*}\bar{\theta}_2 &= -u_{23}^*\bar{u}_*^{12}\bar{\theta}_1 + u_{13}^*\bar{u}_*^{12}\bar{\theta}_2 = -\frac{1}{R}\left(\langle w_3 v^1\rangle\bar{\theta}_1 + \langle w_3 v^2\rangle\bar{\theta}_2\right) \\
&= -\frac{1}{2\langle 12\rangle}\left[\langle w_3|(|2\rangle - |1\rangle)(\eta_2+\eta_1) + \langle w_3|(|4\rangle - |3\rangle)(\eta_4+\eta_3)\right] \\
&= -\frac{1}{2\langle 12\rangle}\langle w_3|(|2\rangle\eta_1 - |1\rangle\eta_2 + |4\rangle\eta_3 - |3\rangle\eta_4) \\
&= \epsilon(-\alpha_+\eta_4 + \beta_+\eta_1)\,, \qquad (2.161)
\end{aligned}$$

where I used (super)-momentum conservation

$$-|1\rangle\eta_1 + |2\rangle\eta_2 - |3\rangle\eta_3 + |4\rangle\eta_4 = \mathcal{O}(\epsilon)\,, \quad \langle 14\rangle = \langle 23\rangle + \mathcal{O}(\epsilon^2)\,,$$

and Schouten identities.

Consider the $\delta^6(Q_6)$ factor. It is better to start from the conjecture. From the recursion relation result, $\mathcal{A}_4$ part (2.144) gives the super-momentum conservation with soft correction like

$$\begin{aligned}
\tilde{Q}_4 &\equiv -|1\rangle(\eta_1 + \epsilon\beta_+\eta_6) + |2\rangle\eta_2 - |3\rangle\eta_3 + |4\rangle(\eta_4 - \epsilon\alpha_+\eta_5) \\
&= -|1\rangle\left(\eta_1 + \epsilon\frac{\langle 45\rangle + \langle 46\rangle}{\langle 14\rangle}\eta_6\right) + |2\rangle\eta_2 - |3\rangle\eta_3 + |4\rangle\left(\eta_4 - \epsilon\frac{\langle 15\rangle + \langle 16\rangle}{\langle 14\rangle}\eta_5\right) \\
&= Q_4 - \frac{\epsilon}{\langle 14\rangle}\left(\langle 15\rangle|4\rangle\eta_5 + \langle 46\rangle|1\rangle\eta_6 + \langle 16\rangle|4\rangle\eta_5 + \langle 45\rangle|1\rangle\eta_6\right) \\
&= Q_4 - \frac{\epsilon}{\langle 14\rangle}\left(\left(\langle 14\rangle|5\rangle + \langle 45\rangle|1\rangle\right)\eta_5 - \left(\langle 14\rangle|6\rangle - \langle 16\rangle|4\rangle\right)\eta_6 + \langle 16\rangle|4\rangle\eta_5 + \langle 45\rangle|1\rangle\eta_6\right) \\
&= Q_6 - \epsilon\frac{\langle 45\rangle}{\langle 14\rangle}|1\rangle(\eta_5+\eta_6) - \epsilon\frac{\langle 16\rangle}{\langle 14\rangle}|4\rangle(\eta_5+\eta_6) \\
&= Q_6\,. \qquad (2.162)
\end{aligned}$$

The last equality holds on the support of (2.160). If one think above equality conversely, the six-point supermomentum conservation becomes the four-point supermomentum conservation with soft correction under the double soft limit.

Finally, the 6-point amplitude with λ-parity operator becomes

$$\begin{aligned}
\mathcal{A}_6|_{\epsilon\to 0} &= (1+\pi)\left(\frac{32\delta^3(P)\delta^6(Q_6)\delta(\zeta_+)}{R^3 M_1^+ M_2^+ M_3^+}\right) \\
&= (1+\pi)\left(\frac{\delta^3(\bar{\theta}_3+\epsilon(-\alpha_+\eta_4+\beta_+\eta_1))}{2\epsilon^2\langle 14\rangle\alpha_+\beta_+}\frac{\delta^3(P)\delta^6(\hat{Q}_4)}{\langle 12\rangle\langle 14\rangle}\right) \\
&= \left(\frac{1}{\epsilon^2}S^{(0)}+\frac{1}{\epsilon}S^{(1)}\right)\mathcal{A}_4\,, \qquad (2.163)
\end{aligned}$$

if one expand the second limit in terms of ϵ up to leading and sub-leading orders. As a result, I confirm the 6-point result using the soft theorem or vice versa.

8-point amplitude soft limit

Due to the computational complexity, I will check only leading order soft theorem in the 8-point amplitude.

In the previous subsection, I didn't mention about the explicit u-gauged C-matrix of the 8-point amplitudes.[4] To study the double soft limit of the 8-point amplitude, one need explicit result of the 8-point amplitude or its constituents. For this reason, I choose the u-cyclic gauge. The C-matrix is given by

$$C=\begin{pmatrix} 1 & 1 & -\hat{u}_{12} & \hat{u}_{12} & -\hat{u}_{13} & \hat{u}_{13} & -\hat{u}_{14} & \hat{u}_{14} \\ \hat{u}_{12} & -\hat{u}_{12} & 1 & 1 & -\hat{u}_{23} & \hat{u}_{23} & -\hat{u}_{24} & \hat{u}_{24} \\ \hat{u}_{13} & -\hat{u}_{13} & \hat{u}_{23} & -\hat{u}_{23} & 1 & 1 & -\hat{u}_{34} & \hat{u}_{34} \\ \hat{u}_{14} & -\hat{u}_{14} & \hat{u}_{24} & -\hat{u}_{24} & \hat{u}_{34} & -\hat{u}_{34} & 1 & 1 \end{pmatrix}\,, \quad \hat{u}_{mn}=\hat{u}_{mn}(z)\,. \qquad (2.164)$$

[4]As explained, u-gauge has many branches like u-cyclic gauge, u-factorization gauge, etc. Explicit result of the 8-point amplitude does not depend on the u-gauge choice.

In this gauge, the fermionic delta function reduces to

$$
\begin{aligned}
\delta^{12}(C\cdot\eta) &= \prod_{I=1}^{3}\frac{1}{4!}\epsilon^{mnpq}(\bar\theta^I_m+\hat u_{mi}(z)\theta^{iI})(\bar\theta^I_n+\hat u_{nj}(z)\theta^{jI})(\bar\theta^I_p+\hat u_{pk}(z)\theta^{kI})(\bar\theta^I_q+\hat u_{ql}(z)\theta^{lI}) \\
&= J^F_8\,\delta^6(Q)\prod_{I=1}^{3}(A_Iz^2+B_Iz+C_I)\,, \quad J^F_8=\left(\frac{4}{R}\right)^3 . \qquad (2.165)
\end{aligned}
$$

The explicit form of fermionic bilinear coefficients are

$$
A_I=-\frac{1}{4}\epsilon_{mnpq}\bar u_*^{mn}\theta^{pI}\theta^{qI}\,, \quad B_I=\frac{1}{4}\epsilon^{mnpq}\epsilon_{prxy}\bar u^*_{mn}\bar u_*^{xy}\theta^{rI}\bar\theta^I_q\,, \quad C_I=\frac{1}{4}\epsilon^{mnpq}u^*_{mn}\bar\theta^I_p\bar\theta^I_q\,. \qquad (2.166)
$$

The minors appear in contour integral formula can be easily read from C-matrix. For example, first minor $M_1(z)$ is obtained by

$$
M_1(z)\equiv a_1z^2+b_1z+c_1=4(\bar u_*^{12}\bar u_*^{34}z^2+(u^*_{12}\bar u_*^{12}+u^*_{34}\bar u_*^{34}+1)z+u^*_{12}u^*_{34})\,, \qquad (2.167)
$$

where I used identities (2.90). The other minors are similar with this.

In order to study the soft limit more clearly, I choose the contours $\mathcal{C}_2$ and $\mathcal{C}_4$ rather than $\mathcal{C}_1$ and $\mathcal{C}_3$ which was used in the previous section. Of course, the two choices are equal up to an overall sign, but $\mathcal{C}_2$ and $\mathcal{C}_4$ contours choice shows much nice soft behavior. The result is very similar to the previous contour choice

$$
\mathcal{A}_8=\delta^3(P)\delta^6(Q)(1+\pi)J^B_8J^F_8\left(\frac{F(2)}{\Delta_{21}\Delta_{23}\Delta_{24}}+\frac{F(4)}{\Delta_{41}\Delta_{42}\Delta_{43}}\right)\,, \quad J^B_8=\frac{1}{2R}\,. \qquad (2.168)
$$

Again the λ-parity operator π acts on $\mathcal{A}_8$ as

$$
\pi:\Lambda_8\to-\Lambda_8\,. \qquad (2.169)
$$

The numerators $F(2)$ and $F(4)$ are given by

$$F(2) = -K_{22} J_{21}{}^{(5} J_{23}{}^{6} J_{24}{}^{7)} + \frac{3}{4} L^{2}{}_{(1}{}^{2}{}_{3} J_{4)2(5} L^{2}{}_{6}{}^{2}{}_{7)} \,, \tag{2.170}$$

$$F(4) = -K_{44} J_{41}{}^{(5} J_{43}{}^{6} J_{42}{}^{7)} + \frac{3}{4} L^{4}{}_{(1}{}^{4}{}_{3} J_{2)4(5} L^{4}{}_{6}{}^{4}{}_{7)} \,, \tag{2.171}$$

where K, J, L are defined in the (2.114). I already obtained the physical poles in the previous section, and they are related to the Δ_{ij} factors through

$$\begin{aligned} \Delta_{21} &= -\frac{2^{10}}{R^4} p_{234}^2 \, p_{678}^2 \,, \quad \Delta_{23} = -\frac{2^{10}}{R^4} p_{345}^2 \, p_{781}^2 \,, \\ \Delta_{41} &= -\frac{2^{10}}{R^4} p_{123}^2 \, p_{567}^2 \,, \quad \Delta_{43} = -\frac{2^{10}}{R^4} p_{456}^2 \, p_{812}^2 \,. \end{aligned} \tag{2.172}$$

The factor Δ_{24} corresponds to spurious poles.

One can see that soft divergent terms come from Δ_{21} and Δ_{23} when one takes particles 7 and 8 become soft limit, and they are only appearing in $\mathcal{C}_2$ contour in which picks the residues of $M_2(z)$. This is the reason why I use $\mathcal{C}_2$ and $\mathcal{C}_4$ contours instead of $\mathcal{C}_1$ and $\mathcal{C}_3$ in which the two divergent contributions are divided into two different on-shell diagrams.

Likewise 6-point case, the double soft limit of 7 and 8 in the light-cone basis is realized by

$$w_4 \to \epsilon \, w_4 \,, \quad v^4 \to \epsilon \, v^4 \,. \tag{2.173}$$

In the $\epsilon \to 0$ limit, u^*_{m4} and $\bar{u}^{n4}_*$ are of order ϵ. As I discussed earlier, kinematic invariants receive ϵ^2 corrections, so one can freely use the 6-point amplitude kinematic relations. For example, the identity (2.53) in the soft limit implies that

$$1 + u^*_{12} \bar{u}^{12}_* + u^*_{23} \bar{u}^{23}_* + u^*_{13} \bar{u}^{13}_* = 0 + \mathcal{O}(\epsilon^2) \,. \tag{2.174}$$

So the minors $M_i(z)$ which consist of kinematic invariant u^*_{mn} and $\bar{u}^{mn}_*$ are directly reduced to the leading order part. To the leading order in ϵ, the all coefficients of minors $M_i(z) = a_i z^2 + b_i z + c_i$ are reduced to the following

$$
\begin{aligned}
&a_1 = 4\epsilon \bar{u}^{12}_* \bar{u}^{34}_* \,, \quad b_1 = 4(\bar{u}^{12}_* u^*_{12} + 1)\,, \quad c_1 = 4\epsilon u^*_{12} u^*_{34}\,, \\
&a_2 = 2\epsilon(-\bar{u}^{24}_* + \bar{u}^{23}_* \bar{u}^{34}_* + \bar{u}^{12}_* \bar{u}^{14}_*)\,, \quad b_2 = 2(u^*_{12}\bar{u}^{23}_* + u^*_{23}\bar{u}^{12}_* + u^*_{13} + \bar{u}^{13}_*)\,, \\
&\qquad c_2 = 2\epsilon(-u^*_{24} + u^*_{23}u^*_{34} + u^*_{12}u^*_{14})\,, \\
&a_3 = 4\epsilon \bar{u}^{14}_* \bar{u}^{23}_* \,, \quad b_3 = 4(u^*_{23}\bar{u}^{23}_* + 1)\,, \quad c_3 = 4\epsilon u^*_{14}u^*_{23}\,, \\
&a_4 = 2(-\bar{u}^{13}_* + \bar{u}^{12}_*\bar{u}^{23}_*)\,, \quad b_4 = 2\epsilon(u^*_{12}\bar{u}^{14}_* + u^*_{23}\bar{u}^{34}_* + u^*_{34}\bar{u}^{23}_* + u^*_{14}\bar{u}^{12}_* + u^*_{24} + \bar{u}^{24}_*)\,, \\
&\qquad c_4 = 2(-u^*_{13} + u^*_{12}u^*_{23})\,.
\end{aligned} \tag{2.175}
$$

Because I am only focusing on the leading order in ϵ, the supermomentum conserving delta function of $\mathcal{A}_8$ trivially reduces to that of $\mathcal{A}_6$

$$
\delta^6(Q_8)|_{\epsilon\to 0} = \delta^6(Q_6)\,. \tag{2.176}
$$

The non-vanshing contribution in the numerator $F(2)$ is

$$
F(2)|_{\epsilon\to 0} = -(b_2)^3 c_4 \alpha_{12}\alpha_{23}\; a_5 a_6 a_7 + (b_2)^3 a_4 \gamma_{12}\gamma_{23}\; c_5 c_6 c_7\,. \tag{2.177}
$$

One can easily check that fermionic bilinears $a_{4+I} = A_I$ and $c_{4+I} = C_I$ become

$$
A_I|_{\epsilon\to 0} = -\frac{1}{2}\epsilon_{mnp}\bar{u}^{mn}_* \theta^{pI}\theta^{4I} = -\zeta^I_- \times \theta^{4I}\,, \tag{2.178}
$$

$$
C_I|_{\epsilon\to 0} = \frac{1}{2}\epsilon^{mnp} u^*_{mn} \bar{\theta}^I_p \bar{\theta}^I_4 = \zeta^I_+ \times \bar{\theta}^I_4\,. \tag{2.179}
$$

The soft limit of Δ_{ij} gives

$$
\Delta_{21} = -\alpha_{12}\gamma_{12}\,, \quad \Delta_{23} = -\alpha_{23}\gamma_{23}\,, \quad \Delta_{24} = (b_2)^2 a_4 c_4\,. \tag{2.180}
$$

I observe that the following useful identities hold in the soft limit

$$a_4 = -M_1^- \,, \tag{2.181}$$

$$c_4 = -M_1^+ \,, \tag{2.182}$$

$$\alpha_{12} = -8\epsilon \frac{\langle 67\rangle + \langle 68\rangle}{R} \times M_2^+ = -8\epsilon\beta_+ \frac{\langle 16\rangle}{R} M_2^+ \,, \tag{2.183}$$

$$\gamma_{12} = -8\epsilon \frac{\langle 67\rangle - \langle 68\rangle}{R} \times M_2^- = -8\epsilon\beta_- \frac{\langle 16\rangle}{R} M_2^- \,, \tag{2.184}$$

$$\alpha_{23} = 8\epsilon \frac{\langle 17\rangle + \langle 18\rangle}{R} \times M_3^+ = 8\epsilon\alpha_+ \frac{\langle 16\rangle}{R} M_3^+ \,, \tag{2.185}$$

$$\gamma_{23} = 8\epsilon \frac{\langle 17\rangle - \langle 18\rangle}{R} \times M_3^- = 8\epsilon\alpha_- \frac{\langle 16\rangle}{R} M_3^- \,, \tag{2.186}$$

where $M_i^\pm$ here denote consecutive minors of $C^\pm$ which appear in the 6-point amplitude $\mathcal{A}_6$.

After combining all ingredient, I can confirm the double soft limit 8-point amplitude in the leading order

$$\begin{aligned}
\mathcal{A}_8|_{\epsilon\to 0} &= (1+\pi)\frac{2^5}{R^4}\frac{F(2)}{\Delta_{21}\Delta_{23}\Delta_{24}} \\
&= (1+\pi)\left(\frac{2^5}{R^4}\right)\frac{-(b_2)^3 c_4\alpha_{12}\alpha_{23}\; a_5a_6a_7 + (b_2)^3 a_4\gamma_{12}\gamma_{23}\; c_5c_6c_7}{(b_2)^2 a_4c_4\alpha_{12}\alpha_{23}\gamma_{12}\gamma_{23}} \\
&= (1+\pi)\left(\frac{2^5 b_2}{R^4}\right)\left(-\frac{1}{a_4\gamma_{12}\gamma_{23}}a_5a_6a_7 + \frac{1}{c_4\alpha_{12}\alpha_{23}}c_5c_6c_7\right) \\
&= (1+\pi)\left(\frac{b_2R}{32\langle 16\rangle}\right)\left[\left(\frac{\delta^3(\theta_4)}{2\epsilon^2\langle 16\rangle\alpha_-\beta_-}\right)\frac{32\delta^3(\zeta_-)}{R^3M_1^-M_2^-M_3^-} + \left\{(+)\leftrightarrow(-), \theta\leftrightarrow\bar\theta\right\}\right] \\
&= \frac{1}{\epsilon^2}S^{(0)}\mathcal{A}_6 + \mathcal{O}(1/\epsilon)\,.
\end{aligned} \tag{2.187}$$

In the final step, I used the following non-trivial identity

$$\begin{aligned}
\frac{b_2R}{\langle 16\rangle} &= \frac{2R}{\langle 16\rangle}\left(u_{12}^*\bar u_*^{23} + u_{23}^*\bar u_*^{12} + u_{13}^* + \bar u_*^{13}\right) \\
&= \frac{2}{\langle 16\rangle}\left(\langle w_1v^3\rangle - \langle v^1w_3\rangle + \langle w_1w_3\rangle - \langle v^1v^3\rangle\right) = 8\,.
\end{aligned} \tag{2.188}$$

2.5 Conclusion

I have studied ABJM scattering amplitudes. Although the dynamics of 3d is different with the higher-dimensions one, it shares many interesting properties with $\mathcal{N} = 4$ super-Yang-Mills theory in 4d. I observed the existence of the recursion relation, Grassmannian, soft theorem, etc. in 3d. I study the 4,6,8-point ABJM amplitudes via Grassmannian formula. I develop the systematic tool for Grassmannian formula computation called u-gauge. Using such gauge choice, I compute 8-point ABJM superamplitude for a first time. I observe the physical and spurious poles as expected from on-shell diagram.

I also derive the ABJM double soft theorem which studies the vanishing momenta limit of external particles. Because only even-point amplitudes are well-defined in ABJM theory, I study the double soft limit. I prove the ABJM double soft theorem using the 3d recursion relation and test it for all known results. The ABJM double soft theorem can be used to confirm higher-point amplitude results.

I have borrowed many interesting idea from 4d amplitude business. But the crucial difference between 4d and 3d is the existence of twistors. In 4d, one can do the half-Fourier transform of the one spinor in momentum spinors $(\lambda^{\alpha}, \tilde{\lambda}^{\dot{\alpha}})$, and then one can obtain new variable $(\tilde{\mu}^{\alpha} = i\frac{\partial}{\partial \lambda^{\alpha}}, \tilde{\lambda}^{\dot{\alpha}})$ called twistor. But in 3d, I have only one kind of momentum spinor (λ^{α}). It is one of crucial obstacles in borrowing all languages from 4d to 3d. If one can discover the similar concept of the twistor(or momentum twistor) in 3d, it would be an important turning point of ABJM amplitudes.

Chapter 3

Duality Test in Higher-Dimensional SCFTs

3.1 5d $\mathcal{N} = 1$ QFTs

5d gauge theories are non-renormalizable in the perturbative sense. Nonetheless, in 1996, Seiberg figures out that some of 5d gauge theories can have non-trivial UV fixed point. In this section, I'm going to discuss about various aspects of 5d $\mathcal{N} = 1$ quantum field theories and their non-perturbative effect instantons. In 5d $\mathcal{N} = 1$ quantum field theories(QFTs) with gauge group G, there are vector multiplet with adjoint representation for G and hypermultiplets with appropriate representation for G(it can be fundamental, anti-symmetric, etc.). On the Coulomb branch moduli space where vector multiplet scalar ϕ has nonzero VEV, the gauge group G is broken to the $U(1)^r$ where $r = \text{rank}(G)$. The low energy effective theory is determined by prepotential, and Seiberg computed 1-loop exact quantum prepotential in [38, 70]. Quantum prepotential is given by

$$\mathcal{F}(\phi) = \frac{1}{2g_{\rm cl}^2} h_{ij}\phi^i\phi^j + \frac{k}{6} d_{ijk}\phi^i\phi^j\phi^k + \frac{1}{12}\left(\sum_{\alpha\in\text{root}} |\alpha\cdot\phi|^3 - \sum_f \sum_{w\in\mathbf{R}_f} |w\cdot\phi + m_f|^3\right) , \tag{3.1}$$

where $g_{\rm cl}$ is the classical gauge coupling, $h_{ij} = \text{Tr}(T_iT_j)$, k is the classical Chern-Simons level, and $d_{ijk} = \frac{1}{2}\text{Tr}_{\mathbf{F}}(T_i\{T_j, T_k\})$ where $\mathbf{F}$ means fundamental representation. T_i are the Cartan generators of G. First two terms are classical contribution to the prepotential. Third term comes from integrating out massive W-bosons. Last term comes from integrating out massive hypermultiplets with mass m_f and representation $\mathbf{R}_f$. Coulomb branch metric or effective gauge coupling is given by second derivative of prepotential

$$\tau_{ij} = (g_{\rm eff}^{-2})_{ij} = \frac{\partial^2\mathcal{F}}{\partial\phi^i\partial\phi^j} , \quad ds^2 = \tau_{ij}d\phi^i d\phi^j . \tag{3.2}$$

Seiberg's claim is that quantum correction part of effective coupling should be semi-positive definite on the entire Coulomb space so that when one can take strong coupling limit $g_{cl} \to \infty$ limit, kinetic term remains positive definite. This is a necessary condition for the existence of non-trivial strongly coupled UV-fixed point. In the work [39], Intriligator-Morrison-Seiberg(IMS) classified possible gauge groups and matter contents. IMS said quiver gauge theories are not allowed, so gauge group is only simple gauge groups. Positivity condition also constrains possible representation and the number of hypermultiplets. I will discuss about possible matter contents for $Sp(N)$ and $SU(N)$ gauge groups and explain why quiver gauge theory be ruled out with a simple example.

$SU(N)$ gauge theories

Let's first think about $SU(N)$ gauge theory with N_f fundamental, N_a anti-symmetric, and N_s symmetric hypermultiplets without masses, or one can simply add masses. Under the appropriate Cartan basis for $SU(N)$, the Coulomb branch of the moduli space is given by $\Phi = \mathrm{diag}(a_1, a_2, \cdots, a_N)$ with $\sum_i^N a_i = 0$, modulo the Weyl group action. On the Weyl chamber $a_1 \geq a_2 \geq \cdots \geq a_N$ and strong coupling limit $g_{\mathrm{cl}}^2 \to \infty$, quantum prepotential is given by

$$\mathcal{F} = \frac{1}{12}\left(2\sum_{i<j}^{N}(a_i - a_j)^3 + 2k\sum_{i=1}^{N} a_i^3 - (N_a + N_s)\sum_{i<j}^{N}|a_i + a_j|^3 - (N_f + 8N_s)\sum_{i=1}^{N}|a_i|^3\right). \tag{3.3}$$

As one can see in above equation, matter part always gives negative contribution to the prepotential, so it constrains to possible matter contents. One symmetric matter contributes to the prepotential as much as one anti-symmetric and eight fundamental matters.

Let's consider $SU(3)$ example for simplicity. On the Weyl chamber $a_1 \geq a_2 \geq 0$, effective gauge coupling for $SU(3)$ gauge theory with N_a anti-symmetric matters and N_f fundamental matters is given by

$$(g_{\mathrm{eff}}^{-2})_{ij} = \begin{pmatrix} (10 - N_f - N_a)a_1 + (5 - k - \frac{1}{2}(N_f + N_a))a_2 & 2a_2 + (5 - k - \frac{1}{2}(N_f + N_a))(a_1 + a_2) \\ 2a_2 + (5 - k - \frac{1}{2}(N_f + N_a))(a_1 + a_2) & (7 - k - \frac{1}{2}(N_f + N_a)a_1 + (8 - N_f - N_a)a_2 \end{pmatrix}. \tag{3.4}$$

Note that for $SU(3)$ gauge group fundamental and anti-symmetric matters give same contribution to the prepotential so to the effective gauge coupling. This fact is crucial for later discussion on the instanton partition function computation. The eigenvalues of (3.4) is non-negative when

$$N_f + N_a + 2|k| \leq 6 \,. \tag{3.5}$$

By doing this for general gauge group rank N, IMS found possible matter contents. The general condition is that for all N with $N_a = 0$,

$$N_f \leq 2N - 2|k| \,, \tag{3.6}$$

For $N \leq 8$, $N_a = 1$ and $N_f \leq 8 - N - 2|k|$ also satisfies positivity condition. For $N > 8$, $N_a = 0$ is required. For $N = 4$ case, $N_a = 2$, $N_f = k = 0$ is also possible. This is the IMS bound for $SU(N)$ gauge theories.

$Sp(N)$ gauge theories

The story is parallel with $SU(N)$ gauge theory. The Coulomb branch moduli space is given by $\Phi = \mathrm{diag}(a_1, \cdots, a_N, -a_1, -\cdots, a_N)$ modulo the Weyl group action. On the Weyl chamber $a_1 \geq a_2 \geq \cdots \geq a_N \geq 0$ and strong coupling limit $g_{\mathrm{cl}}^2 \to \infty$, quantum prepotential for $Sp(N)$ gauge theory with N_f fundamental and N_a anti-symmetric matters is given by

$$\mathcal{F} = \frac{1}{6}\left(\sum_{i<j}^{N}\left((a_i - a_j)^3 + (a_i + a_j)^3\right)(1 - N_a) + \sum_{i=1}^{N} a_i^3 (8 - N_f)\right) \,. \tag{3.7}$$

For general N, effective coupling is given by

$$(g_{\mathrm{eff}}^{-2})_{ii} = 2\left((N - i)a_i + \sum_{k=1}^{i-1} a_k\right)(1 - N_a) + a_i(8 - N_f) \,, \tag{3.8}$$

$$(g_{\mathrm{eff}}^{-2})_{i<j} = 2(1 - N_a)a_j \,. \tag{3.9}$$

For $Sp(1) \cong SU(2)$ case, positivity condition for $g_{\rm eff}^{-2} = a_1(8 - N_f)$ requires $N_f \leq 7$. IMS bound for general $Sp(N)$ gauge group is that

$$N_f \leq 7 \quad \text{for} \quad N_a = 1\,, \tag{3.10}$$

$$N_f \leq 2N + 4 \quad \text{for} \quad N_a = 0. \tag{3.11}$$

Other gauge theories

One can find similar stories for $G = SO(N)\,, G_2\,, F_4\,, E_6\,, E_7\,, E_8$ gauge theories in [39]. The list in below is the positivity conditions for $SO(N)$, G_2, F_4, E_6, E_7, and E_8 gauge theories to have non-trivial UV-fixed point.

- $SO(N) : N_{\rm v} \leq N - 4$ vector representation hypermultiplets,
- $G_2 : n_7 \leq 4$ fundamental hypermultiplets,
- $F_4 : n_{26} \leq 3$ fundamental hypermultiplets
- $E_6 : n_{27} \leq 4$ fundamental hypermultiplets,
- $E_7 : n_{56} \leq 3$ fundamental hypermultiplets (can be half-integer),
- E_8 : no possible hypermultiplets.

Quiver gauge theory

Quiver gauge theories such as the gauge theory with $G_1 \times G_2$ gauge group are ruled out, because one can always find negative region for effective couplings in the Coulomb branch. For instance, let's consider $SU(2)\times SU(2)$ quiver gauge theory. There are two scalar fields ϕ_1 and ϕ_2 from two $SU(2)$ gauge multiplets, which are Coulomb branch parameters. In the Weyl chamber $\phi_1 \geq \phi_2 \geq 0$, quantum prepotential for this quiver gauge theory is given by

$$\mathcal{F} = \frac{1}{2g_1^2}\phi_1^2 + \frac{1}{2g_2^2}\phi_1^2 + \frac{1}{6}\left(8\phi^3 + 8\phi_2^3 - (\phi_1+\phi_2)^3 - (\phi_1-\phi_2)^3\right)\,, \tag{3.12}$$

where last two terms come from massless bi-fundamental matter. So the effective gauge coupling under the strong coupling limit $g_{1,2}^2 \to \infty$ is given by

$$g_{\text{eff}}^{-2} = \begin{pmatrix} 6\phi_1 & -2\phi_2 \\ -2\phi_2 & -2\phi_1 + 8\phi_2 \end{pmatrix}, \tag{3.13}$$

and it takes negative eigenvalue around $\phi_2 \sim 0.255\phi_1$. So one can rule out quiver gauge theories for similar reason.

Beyond IMS bound

Recently many 5d QFT examples which exceed IMS bound [48–58] have been found. It implies that there should be a loophole in the IMS argument. Even the existence of quiver gauge theory was found in [40]. The existence of 5d gauge theories beyond IMS bound is beautifully explained in the very recently work by Jefferson, Kim, Vafa, and Zafrir in [59]. I would like to sketch how they can explain the existence of the theories beyond IMS bound. Previous argument based on the semi-positive definiteness of the Hessian of quantum correction part of prepotential(or effective gauge coupling) on the entire Coulomb branch moduli space. But JKVZ say "IMS argument is too strong!". Their claim is that *unphysical* regions can exist on the Coulomb branch moduli space where new massless degrees of freedom can appear. For instance, above $SU(2) \times SU(2)$ quiver gauge theory has such region $\phi_2 = \frac{1}{2}\phi_1$ where instantons become massless. For theories which contain unphysical region, perturbative intuition breakdown. So they relax the positivity condition by requiring that the Hessian of quantum prepotential is semi-positive definite on the only *physical* region of the Coulomb branch moduli space. This is the key idea of the work in [59]. By inspecting Coulomb branch moduli spaces carefully, they classified all possible

5d gauge theories that could have non-trivial UV-fixed point for single gauge node cases. A lot of new 5d gauge theories obtained in their work. Here are a few examples of new possible 5d gauge theories beyond IMS bound

- $SU(N+1) : N_f \leq 2N+6$ fundamental hypermultiplets,
- $Sp(N) : N_f \leq 2N+6$ fundamental hypermultiplets,

where previous bounds were $N_f \leq 2N+2$ for $SU(N+1)$ and $N_f \leq 2N+4$ for $Sp(N)$.

In the thesis, I will focus on the above two 5d gauge theories which saturate the new bound, i.e. $N_f = 2N+6$ for both $SU(N+1)$ and $Sp(N)$. Because they are conjectured to have same UV-fixed point, and this non-trivial UV SCFT is not a 5d SCFT but a 6d SCFT on a circle [55]. Actually, this kind of circle uplift is not a new physical phenomenon. Maximal 5d SYM also has shown this feature as 6d $\mathcal{N}=(2,0)$ on a circle [61,62], and index computation [65–67] which counts BPS bound states confirms this relation [71–73]. 5d SYM description for circle reduction of 6d E-string theory [63] also have shown this feature [64]. Above index computations confirm that KK modes along the 6d circle are mapped to the instantons. Instantons are playing a crucial role in the duality. 5d instantons are solitonic particles which satisfy self-dual Yang-Mills equation $F_{\mu\nu} = (*F)_{\mu\nu}$ on spatial $\mathbb{R}^4 \in \mathbb{R}^{4,1}$. I will discuss about the instanton in the next subsection.

Nekrasov partition function

A physical observable that I want to study is Nekrasov partition function for 5d $SU(3)$ and $Sp(2)$ theories.[1] There are eight real supercharges in 5d $\mathcal{N}=1$ gauge theory. The eight supercharges are decomposed into Q^A_α, $\bar{Q}^B_{\dot{\beta}}$, where

[1] In later, I will also discuss about $Sp(N)$ generalization.

A, B are doublet indices of $SU(2)_R$ symmetry and $\alpha, \dot{\beta}$ are doublet indices of $SU(2)_l \times SU(2)_r = SO(4)$ rotation of spatial $\mathbb{R}^4 \in \mathbb{R}^4 \times \mathbb{R}^1$. 5d $\mathcal{N} = 1$ SUSY algebra is given by

$$\{Q_M^A, Q_N^B\} = P_\mu(\Gamma^\mu C)_{MN}\epsilon^{AB} + i\frac{4\pi^2 k}{g_0^2} C_{MN}\epsilon^{AB} + i\mathrm{tr}\,(v\Pi)\, C_{MN}\epsilon^{AB}\,, \quad (3.14)$$

where M, N are Dirac indices, C_{MN} is charge conjugation matrix, k is instanton number, α_i are Coulomb VEVs and Π_i are electric charges. Instantons can make marginal bound states with charged W-bosons and their superpartners with the following BPS mass

$$M = \frac{4\pi^2 k}{g_0^2} + \mathrm{tr}\,(v\Pi)\ , \quad (3.15)$$

and they preserve half of the SUSY. Nekrasov partition function is an index which counts the marginal bound states of BPS particles. The definition of Nekrasov partition function is as follows

$$Z_{\mathrm{Nek}} = \mathrm{Tr}\left[(-1)^F q^k e^{-\beta\{Q,Q^\dagger\}} e^{-\epsilon_1(J_1+J_R)} e^{-\epsilon_2(J_2+J_R)} e^{-\mathrm{tr}(\alpha\Pi)} e^{-m\cdot F}\right]\,, \quad (3.16)$$

where Tr be taken over 5d QFT Hilbert space, $Q = \bar{Q}^1_{\dot{1}}$ and $Q^\dagger = -\bar{Q}^2_{\dot{2}}$, $q = \exp\left(-\frac{4\pi^2}{g_0^2}\right)$, J_R, J_1, J_2 are Cartans of $SU(2)_R$ and $SO(4)$ with omega deformation parameters $\epsilon_{1,2}$, and one can add flavor chemical potentials m_l for global symmetry Cartans F_l. Nekrasov partition function can be decomposed into the perturbative and non-perturbative(instanton) parts

$$Z_{\mathrm{Nek}} = Z_{\mathrm{pert}} Z_{\mathrm{inst}} = Z_{\mathrm{pert}}(1 + \sum_{k=1}^{\infty} q^k Z^k_{\mathrm{inst}})\,. \quad (3.17)$$

The main interest of this thesis is how to compute this instanton partition function in the presence of too many hypermultiplets. Instanton partition function can be computed by Witten index of ADHM instanton quantum mechanical system. So one needs to study ADHM instanton QM or its stringy ADHM gauged QM.

3.1.1 Instantons and ADHM quantum mechanics

In this subsection, I will discuss about instantons, which are playing a crucial role in 5d gauge theories in the strong coupling limit. Since their mass is proportional to the inverse gauge coupling $m_0 \sim g_0^{-2}$, they become massless in strong coupling limit. Therefore, one has to consider this non-perturbative objects in the strongly coupled CFT. This instanton review is based on the David Tong's lecture note on solitons [74], and their index review is based on the recent work on the 5d instanton counting [64].

Definition of Instanton

Let's first consider 4d Yang-Mills instanton. For a given 4d $SU(N)$ Euclidean Yang-Mills action

$$S_{\rm E} = \frac{1}{2g_0^2}\int d^4x \; {\rm Tr}\; F_{\mu\nu}F^{\mu\nu}\,, \tag{3.18}$$

instantons are solutions of classical Yang-Mills equation of motion

$$D_\mu F^{\mu\nu} = 0\,. \tag{3.19}$$

Therefore, they will give the finite action configuration. Finiteness requires it must be pure gauge at boundary of spatial infinity in $\mathbb{R}^4$

$$A_\mu = ig^{-1}\partial_\mu g\,, \tag{3.20}$$

where $g(x) \in SU(N)$ group element. This instanton configuration gives non-trivial map from spatial infinity S^3_∞ to the $SU(N)$, and it is given by third homotopy group

$$\Pi_3(SU(N)) \cong \mathbb{Z}\,. \tag{3.21}$$

For instance, the gauge group $SU(2)$ is equivalent with manifold $SU(2) \cong S^3$, and then $k \in \mathbb{Z}$ is the winding number. One can define integral-valued instanton number k(winding number), which is given by

$$\begin{aligned} k &= \frac{1}{16\pi^2}\int d^4x \mathrm{Tr}\left(\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) \\ &= \frac{1}{24\pi^2}\int_{S_\infty} d^3S_\mu \mathrm{Tr}\left(\epsilon^{\mu\nu\rho\sigma}(\partial_\nu g)g^{-1}(\partial_\rho g)g^{-1}(\partial_\sigma g)g^{-1}\right) \in \mathbb{Z}\,. \end{aligned} \tag{3.22}$$

There is another way to obtain first order instanton equation rather than second order equation in (3.19). One can find following inequality for Yang-Mills action

$$\begin{aligned} S_{\mathrm{E}} &= \frac{1}{2g_0^2}\int d^4x \ \mathrm{Tr}\ F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{4g_0^2}\int d^4x \ \mathrm{Tr}\ (F_{\mu\nu} \mp *F_{\mu\nu})^2 \pm 2F_{\mu\nu} * F^{\mu\nu} \\ &\geq \pm\frac{1}{2g_0^2}\int d^4x \ \mathrm{Tr}\ F_{\mu\nu} * F^{\mu\nu} \\ &= \frac{8\pi^2}{g_0^2}k\,. \end{aligned} \tag{3.23}$$

Self-dual(or anti-self-dual) condition saturates the inequality

$$F_{\mu\nu} = \pm\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,, \tag{3.24}$$

so it is the configuration which minimize the action. This is called self-dual instanton equation. The self-dual instanton equation also satisfies Yang-Mills equation of motion

$$D_\mu F^{\mu\nu} = D_\mu(*F)^{\mu\nu} = 0\,, \tag{3.25}$$

by the Bianchi identity. Now the problem is translated into how to find solutions of self-dual YM equation $F = *F$.

One instanton solution for $SU(2)$ theory

One example is the $k = 1$ instanton solution for $SU(2)$ YM theory in the singular gauge

$$A^{\mu} = \frac{\rho^2 (x - X)_{\nu}}{(x - X)^2)\left((x - X)^2 + \rho^2\right)} \bar{\eta}^i_{\mu\nu} \left(g \sigma^i g^{-1}\right) \,. \tag{3.26}$$

The $\bar{\eta}^i$ are anti-symmetric 't Hooft matrices

$$\bar{\eta}^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \quad \bar{\eta}^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \quad \bar{\eta}^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{3.27}$$

The σ^i are Pauli matrices which carry $SU(2)$ gauge indices. This one-instanton solution has eight parameters, and they are called *collective coordinates*

- 4 translations X^{μ} : Localized objection in $\mathbb{R}^4$
- 1 scale ρ : Instanton size
- 3 global gauge transformations $g \in SU(2)$.

For general k, there are $8k$ number of collective coordinates. One important thing is that as $\rho \to 0$ instanton solution has singularity at $x^{\mu} = X^{\mu}$, and it is called a small instanton singularity.

ADHM construction

Now I will briefly discuss about how to construct the k self-dual instanton solution for the $SU(N)$ gauge theory called ADHM construction [75]. One can think $k \times k$ complex matrices Z_1, Z_2 and a $k \times N$ complex matrix ψ and a $N \times k$ complex matrix $\tilde{\psi}$. They are called ADHM data and satisfy following ADHM constraint equations

$$\mu_r = [Z_1, Z_1^\dagger] + [Z_2, Z_2^\dagger] + II^\dagger - J^\dagger J = 0\,, \tag{3.28}$$

$$\mu_c = [Z_1, Z_2] + IJ = 0\,. \tag{3.29}$$

Using the Pauli matrices $\sigma^m_{\alpha\dot{\beta}}$, one can rewrite space time coordinates x_m as following

$$x_{\alpha\dot{\beta}} = \sigma^m_{\alpha\dot{\beta}} x_m \equiv \begin{pmatrix} z_2 & z_1 \\ -\bar{z}_1 & \bar{z}_2 \end{pmatrix}\,. \tag{3.30}$$

One can consider the following $(N + 2k) \times 2k$ matrix

$$\Delta = \begin{pmatrix} I & J \\ Z_2 + z_2 & Z_1 + z_1 \\ -Z_1^\dagger - \bar{z}_1 & Z_2^\dagger - \bar{z}_2 \end{pmatrix}\,. \tag{3.31}$$

The conditions $\mu_r = \mu_c = 0$ lead to the following identity

$$\Delta^\dagger \Delta = \begin{pmatrix} f^{-1} & 0 \\ 0 & f^{-1} \end{pmatrix}\,, \tag{3.32}$$

where $f(x)$ is $k \times k$ hermitian matrix. The matrix Δ defines $2k$ vectors in $\mathbb{C}^{2k+N}$, so null space is dimension N. One can define $(N + 2k) \times N$ matrix U, which is an orthonormal complement of Δ. So it satisfies $UU^\dagger = 1_N$ and $\Delta^\dagger \cdot U = 0$. It is natural to define following projection operator

$$P = \Delta \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \Delta^\dagger\,, \tag{3.33}$$

so it satisfies $PU = 0,\ P\Delta = \Delta$. Then self-dual solution is given by

$$A_\mu = iU^\dagger \partial_\mu U\,. \tag{3.34}$$

One can simply check the self-dual condition of $F_{\mu\nu} = \partial[\mu A_{\nu]} - i[A_\mu, A_\nu]$. This is the basic idea of ADHM construction.

String theory ADHM construction

In 5d gauge theories, instantons are solitonic particles localized on $\mathbb{R}^4$ which satisfy self-dual condition (3.24) on $\mathbb{R}^4$. 5d gauge theories are non-renormalizable in perturbative sense, and it cause the small instanton singularity in the 1d ADHM instanton quantum mechanics. To resolve this singularity, one has to UV-complete by introducing new UV degrees of freedom. It is accomplished by studying UV-complete D-brane engineered stringy ADHM construction [76–78]. In string theory perspective, ADHM construction appears very naturally. $(p+1)$-dimensional $U(N)$ gauge theory is described by low energy effective theory on N coincident Dp-branes. k instantons are realized by k D$(p-4)$-branes living on Dp-branes. It is confirmed by computing their charges and masses [79, 80]. This instanton configuration(= Dp-D$(p-4)$-brane configuration) preserves half of the SUSY. In the perspective of k D$(p-4)$-branes, low energy theory is given by $(p-3)$-dimensional $U(k)$ gauge theory with adjoint hypermultiplets and N fundamental hypermultiplets, and they correspond with the ADHM data Z_1, Z_2 and I, J. The ADHM constraint equations are nothing but the D-term and F-term conditions in $U(k)$ SUSY gauge theory. The $U(k)$ gauge theory also contains vector multiplets which is desired new UV degrees. Since the instantons are realized by D$(p-4)$-branes living on Dp-branes, Higgs branch where vector multiplet scalars have zero VEV gives correct description for self-dual instantons.

Let's focus on the 1d ADHM gauged quantum mechanical system. For each gauge group G=$U(N)$, $Sp(N)$ of 5d theories, the dual 1d ADHM gauge group is given by $\hat{G}$=$U(k)$, $O(k)$ respectively, where k is an instanton number.[2] ADHM data is given by various hypermultiplet excitations along the D-branes. The 1d ADHM data is given by

$$(q_{\dot{\alpha}}, \psi^A)\,, \quad (a_{\alpha\dot{\beta}} = \frac{1}{\sqrt{2}} a_m \sigma^m_{\alpha\dot{\beta}}, \lambda^A_\alpha)\,, \tag{3.35}$$

where $q_{\dot{\alpha}}$ in the bi-fundamental representation of $G \times \hat{G}$, and a_m in the adjoint, symmetric representation of $\hat{G} = U(k), O(k)$. The indices $\alpha, \dot{\beta}$ are doublet indices of $SU(2)_l \times SU(2)_r = SO(4)$ of $\mathbb{R}^4$ rotation, and A is $SU(2)_R$ symmetry index of the 5d SUSY. These 1d hypermultiplets can be regarded as a circle reduction of 2d $\mathcal{N} = (0,4)$ multiplets. This 1d system preserves half of SUSY with four supercharges $\bar{Q}^A_{\dot{\alpha}}$, so the R-symmetry is $SU(2)_r \times SU(2)_R = SO(4)$. One has to introduce an additional degree to obtain UV-complete ADHM description, and it is a vector multiplet which consists of a gauge field A_t, a real scalar ϕ, fermions $\lambda^A_{\dot{\alpha}}$ and 3 auxiliary field $D_{\dot{\alpha}\dot{\beta}} = D_{\dot{\beta}\dot{\alpha}}$. If one couples matter fields in 5d gauge theory with various representations on G, they give new UV degrees of freedom in 1d ADHM gauged QM. Brane configuration will be useful to find this degrees.

Witten index of ADHM QM

I'm interested in the instanton partition function of 5d $\mathcal{N} = 1$ gauge theories, and it was discussed in the recent work [64]. I will follow their idea for computing

[2] I won't consider $SO(N)$ gauge group, but it has same story.

instanton partition functions with many flavors. Instanton part of Nekrasov partition function in (3.17) can be computed by Witten index of the 1d ADHM gauged quantum mechanics

$$Z_{\text{inst}}^{k}(\epsilon_1, \epsilon_2, \alpha_i, m_l) = \text{Tr}\left[(-1)^F e^{-\beta\{Q,Q^\dagger\}} e^{-\epsilon_1(J_1+J_R)} e^{-\epsilon_2(J_2+J_R)} e^{-\alpha_i \cdot \Pi_i} e^{-m_l \cdot F_l}\right], \tag{3.36}$$

One can compute the Witten index exactly using the path integral representation in the weakly coupling regime $(g_{1d}^{2/3}\beta) \to 0$ via localization technique, where g_{1d}^2 is 1d gauge coupling. Finally, this index reduces to contour integrals over gauge zero modes. The 1d quantum mechanical index can be regarded as 1d reduction of elliptic genus of the 2d $\mathcal{N} = (0, 4)$ gauge theory. Recently, 2d $\mathcal{N} = (0, 2)$ gauge theory indices are exactly computed and contour description is obtained by the work in [81–83]. One can compute 1d index by decomposing the $(0, 4)$ matters into $(0, 2)$ language and dimensional reduction. The 1d index is nothing but the change of elliptic theta functions to the hyperbolic functions. Many continuum which are appeared in the index computation can be regulated by introducing various chemical potential $\epsilon_{1,2}$, α_i, m_l, etc. Some of them can be regarded as effective masses of fields or IR regulators. But there is continuum come from the vector multiplet scalars. One can't turn on the effective mass(or chemical potential), since they does not charged under the global symmetry. One needs to treat this very carefully. This is the main difference with 2d elliptic genus where there is no such issue.

This continuum degrees correspond with the Coulomb branch degrees in the ADHM gauge theory, because vector multiplet scalars are related with the distance between two branes. In the D-brane perspective, this system can contain extra Coulomb branch degrees compared with original 5d QFT degrees. At the level of index, one can expect the index can be decomposed into the original QFT degrees and this string theory degrees

$$Z_{\text{ADHM}} = Z_{\text{extra}} Z_{\text{inst}} \,. \tag{3.37}$$

If one want to obtain correct QFT partition function, the extra stringy part should separately be computed in 1d ADHM system.

Back to the index story. After localization computation, one can obtain contour integral formula for the Witten index

$$Z = \frac{1}{|W|} \oint e^{\kappa \, \text{tr}(\phi)} Z_{\text{1-loop}} = \frac{1}{|W|} \oint e^{\kappa tr(\phi)} Z_V \prod_{\Phi} Z_\Phi \prod_{\Psi} Z_\Psi \,, \tag{3.38}$$

where Z_V, Z_Φ, Z_Ψ are the 1-loop determinant for vector, chiral, Fermi multiplets in the 1d ADHM theory. They are depends on the various chemical potential and representions on $G, \hat{G}$. κ is 1d CS coupling. W is the Weyl group of $\hat{G}$. Such 1-loop determinants for various multiplets are given by

$$(0,2) \text{ vector } : \; Z_V = \prod_{\alpha \in \text{root}} 2 \sinh \frac{\alpha(\phi)}{2} \prod_{i=1}^{r} \frac{d\phi_i}{2\pi i} \tag{3.39}$$

$$(0,2) \text{ chiral } : \; Z_\Phi = \prod_{\rho \in R_\Phi} \frac{1}{\sinh\left(\frac{\rho(\phi) + J\epsilon_+ + F \cdot z}{2}\right)} \tag{3.40}$$

$$(0,2) \text{ Fermi } : \; Z_\Psi = \prod_{\rho \in R_\Psi} \sinh\left(\frac{\rho(\phi) + J\epsilon_+ + F \cdot z}{2}\right) \tag{3.41}$$

where R_Φ,R_Ψ are the representations of the chiral and Fermi multiplets for the dual gauge group $\hat{G}$, and ρ is the weights of the representations, and r is the rank of the dual ADHM gauge group $\hat{G}$. F is the collective notation for original gauge symmetry G and other global symmetries, and z are corresponding chemical potentials. $J = \frac{J_1 + J_2}{2} + J_R$ and $\epsilon_\pm = \frac{\epsilon_1 \pm \epsilon_2}{2}$. The index reduces to the r-dimensional complex contour integral of the variables $\phi_i = \varphi_i + iA_t$ which consist of gauge holonomy on a circle A_t and vector multiplet scalar φ_i. Contour description is derived in [81] for 2d theory and in [64] for 1d theory, and it is called Jeffrey-Kirwan residue prescription.

Jeffrey-Kerwin residue

I will explain about the Jeffrey-Kirwan residue(JK residue) shortly [64] . The Witten index given in (3.38) is given by contour integral formula, and simple poles are arisen from the 1-loop determinant of chiral multiplets. The Jeffrey-Kirwan residue gives the correct contour prescription for which poles one should pick. The Jeffrey-Kirwan residue is given by

$$\frac{1}{|W|}\oint Z_{\text{1-loop}} = \frac{1}{|W|}\sum_{\phi_*}\text{JK-res}_{\phi_*}(Q_*,\mathfrak{n})Z_{\text{1-loop}}\,, \tag{3.42}$$

where $\phi_* = (\phi_{1*},\dots,\phi_{r*})$ denote all possible poles. The $\text{JK-res}_{\phi_*}(Q_*,\mathfrak{n})$ are determined by following rules. Near the pole ϕ_*, one can Laurent expand 1-loop determinant, and simple pole contribution is given by

$$\frac{1}{Q_{j_1}(\phi-\phi_*)\dots Q_{j_r}(\phi-\phi_*)}\,. \tag{3.43}$$

Then one can read r set of r-dimensional charge vectors $Q_* = (Q_1,\dots,Q_r)$ for each pole ϕ_*. Then $\text{JK-res}_{\phi_*}(Q_*,\mathfrak{n})$ are given by

$$\begin{aligned}&\text{JK-res}_{\phi_*}(Q_*,\mathfrak{n})\frac{d\phi_1\wedge\cdots\wedge d\phi_r}{Q_{j_1}(\phi-\phi_*)\dots Q_{j_r}(\phi-\phi_*)}\\ &=\begin{cases}|\det(Q_{j_1},\dots,Q_{j_r})|^{-1} & \text{if } \mathfrak{n}\in\text{cone}(Q_{j_1},\dots,Q_{j_r})\\ 0 & \text{otherwise}\end{cases}\,,\end{aligned} \tag{3.44}$$

where $\mathfrak{n}\in\text{cone}(Q_{j_1},\dots,Q_{j_r})$ means that the arbitrary r-dimensional vector $\mathfrak{n}$ is located in the cone which is made by vectors $(Q_{j_1},\dots,Q_{j_r})$, i.e. $\mathfrak{n}$ can be written as sum of Q_i with positive coefficients $\eta = \sum_{i=1}^{r} a_i Q_{j_i}$ with $a_i \geq 0$.

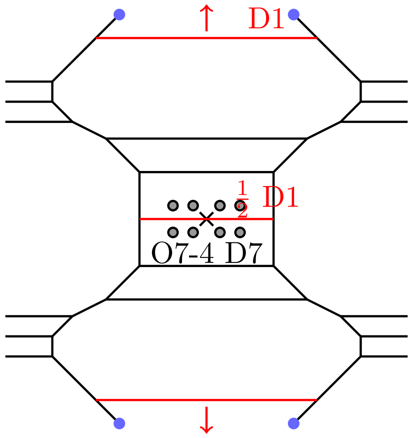
Finally non-trivial values of Jeffrey-Kirwan residue gives correct contour description. Above JK residue only consider poles with finite values. If theory has many 1d Fermi multiplets, it can have pole at infinity. Such 1d Fermi multiplets are arisen from 5d fundamental hypermultiplets degrees. As a result, pole at infinity appears if the theory has too many hypermultiplets. Such pole at infinity correspond with Coulomb branch degrees, which is already discuss in the above. This problem will appear in my 5d gauge theory problems.

3.1.2 Instanton partition function of $Sp(2)$ gauge theory

Let's study the Nekrasov partition function of general 5d $Sp(N+1)$ gauge theory with N_f fundamental hypermultiplets. The 5d Nekrasov partition function consists of the perturbative part and the instanton part $Z_{\text{Nek}} = Z_{\text{pert}} Z_{\text{inst}}$. The 5d instanton partition function for the $Sp(N+1)$ gauge group with matters are well-studied in [64,84]. As I explained above, naive instanton partition functions can contain unwanted degrees freedom, so one should subtract this factor.

I first consider the general 5d $\mathcal{N}=1$ $Sp(N+1)$ gauge theories with $N_f = 2N+8$ fundamental hypermultiplets. Type IIB brane diagram for $N=1$ case is given in Figure 3.1.

Instantons are realized by the D1-branes living on the D5-branes. One should carefully use the string theory engineered ADHM construction. It contains unwanted extra degrees of freedoms that can't be controlled [64]. For example, Figure 3.2 shows the brane diagram for $Sp(N+1)$ gauge theory with $N_f = 2N+6$ matters at $N=1$, which was considered in [38]. In this case D1-branes which can freely escape to infinity without feeling any force, and it provide extra degrees of freedom i.e. Coulomb branch degrees. Their contribu-



	0	1	2	3	4	5	6	7	8	9
D1	•	-	-	-	-	-	•	-	-	-
NS5	•	•	•	•	•	•	-	-	-	-
D5	•	•	•	•	•	-	•	-	-	-
O7-D7	•	•	•	•	•	-	-	•	•	•

Figure 3.1 Type IIB brane diagram for the 5d $\mathcal{N}=1$ $Sp(2)$ gauge theory with $N_f=10$ hypermultiplets. The figure shows the covering space of $\mathbb{Z}_2$ quotient by O7 (the cross in the figure). The blue dots denote 7-branes on which vertical 5-branes can end. Half-D1-brane is stuck to the $\mathrm{O7}^-$-plane.

tion to the instanton partition function can be computed separately. To obtain correct instanton partition function, one should subtract this extra contribution from the ADHM quantum mechanical index like (3.37). However, for 5d $Sp(N+1)$ gauge theory with $N_f=2N+8$ matters, it is unkown how to identify the contribution of the extra degrees of freedom to the index. The extra states are supposed to be provided by the D1-branes moving vertically away from the D5-branes. I currently do not have technical controls of such extra states.[3]

However one-instanton sector is special, because this sector is realized by the half-D1-brane stuck to $\mathrm{O7}^-$-plane. The half-D1-brane can not escape to infinity, so it does not contains any extra degrees, and at the level of index $Z_{\text{extra}}=1$ in (3.37). For this reason, one can study the one-instanton sector

[3]Effective potential for φ is proportional to the $-\log$ of 1-loop determinants: $V(\varphi)=-\log Z_{\text{1-loop}}(\varphi)$. 5d fundamental hypermultiplet contributes 1d Fermi multiplet degrees. As a result, too many hypermultiplets cause the repulsive force to φ. I don't know how to treat such theory.

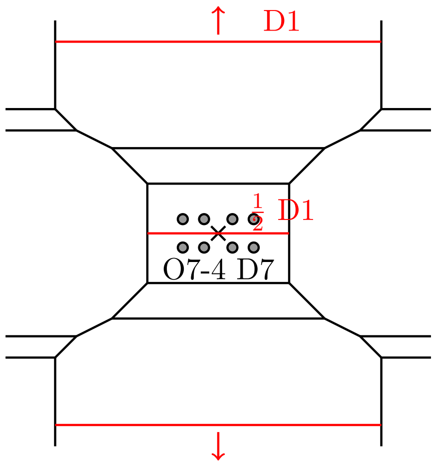


Figure 3.2 Type IIB brane diagram for the 5d $\mathcal{N}=1$ $Sp(2)$ gauge theory with $N_f=8$ hypermultiplets. D1-branes engineer instanton soliton particles

of the general 5d $Sp(N+1)$ gauge theories with $N_f = 2N+8$ fundamental hypermultiplets using the ADHM description.

Witten index of ADHM gauged quantum mechanics for k instantons Z^k_{ADHM} is given by the sum of $Z^k_{\pm}$, because the dual 1d ADHM gauge group $O(k)$ has two disconnected sectors $O(k)^{\pm}$, [4]

$$Z^k_{\text{ADHM}} = \frac{1}{2}(Z^k_{+} + Z^k_{-})\,. \tag{3.45}$$

In the perspective of the 1d ADHM theory, various 5d multiplets provide many 1d degrees of freedom. I won't explain it, but their contributions were already discussed in the many works, e.g. [52, 64].

[4] Actually $Sp(N+1)$ gauge theory has $\mathbb{Z}_2$ valued θ angle because $\pi_4(Sp(N+1)) = \mathbb{Z}_2$, so its index is given by

$$Z^k_{\text{ADHM}} = \begin{cases} \frac{1}{2}(Z^k_{+} + Z^k_{-}) & ,\ \theta = 0 \\ \frac{(-1)^k}{2}(Z^k_{+} - Z^k_{-}) & ,\ \theta = \pi \end{cases}.$$

But in my case, θ is not important. Its effect can be absorbed by redefinition of the flavor chemical potential.

If one set $k = 2n + \chi$ where $\chi = 0$ or 1, $Z^k_\pm$ is given by

$$Z^k_\pm = \frac{1}{|W|} \oint \prod_{I=1}^{n} \frac{d\phi_I}{2\pi i} Z^\pm_{\text{vec}}(\phi, \alpha; \epsilon_{1,2}) \prod Z^\pm_{R_f}(\phi, \alpha, m_l; \epsilon_{1,2}) \,, \tag{3.46}$$

where Weyl factor $|W|$ is given by

$$|W|^{\chi=0}_+ = 2^{n-1} n! \,, \; |W|^{\chi=1}_+ = 2^n n! \,, \; |W|^{\chi=0}_- = 2^{n-1}(n-1)! \,, \; |W|^{\chi=1}_- = 2^n n! \,. \tag{3.47}$$

R_f denotes the representation of hypermultiplet matters, which is the fundamental representation in my case. See [64] for the details. $Z^\pm_{\text{vec}}$ is 5d vector multiplet contribution to the index, and $Z^\pm_{R_f}$ is the 5d fundamental hypermultiplet contribution with mass m_l. I will use the fugacity variables $v_i = e^{-\alpha_i}$, $y_l = e^{-m_l}, t = e^{-\epsilon_+}, u =^{-\epsilon_-}$.

Vector multiplet part for $O(k)^+$ sector is given by

$$\begin{aligned} Z^+_{\text{vec}} = & \left[\frac{1}{2\sinh \frac{\pm\epsilon_- + \epsilon_+}{2} \prod_{i=1}^{N+1} 2\sinh \frac{\pm\alpha_i + \epsilon_+}{2}} \prod_{I=1}^{n} \frac{2\sinh \frac{\pm\phi_I}{2} 2\sinh \frac{\pm\phi_I + 2\epsilon_+}{2}}{2\sinh \frac{\pm\phi_I \pm\epsilon_- + \epsilon_+}{2}} \right]^\chi \\ & \times \prod_{I=1}^{n} \frac{2\sinh \epsilon_+}{2\sinh \frac{\pm\epsilon_- + \epsilon_+}{2} \prod_{i=1}^{N+1} 2\sinh \frac{\pm\phi_I \pm\alpha_i + \epsilon_+}{2}} \\ & \times \frac{\prod_{I>J}^{n} 2\sinh \frac{\pm\phi_I \pm\phi_J}{2} 2\sinh \frac{\pm\phi_I \pm\phi_J + 2\epsilon_+}{2}}{\prod_{I=1}^{n} 2\sinh \frac{\pm 2\phi_I \pm\epsilon_- + \epsilon_+}{2} \prod_{I>J}^{n} 2\sinh \frac{\pm\phi_I \pm\phi_J \pm\epsilon_- + \epsilon_+}{2}} \,, \end{aligned} \tag{3.48}$$

and for $O(k)^-$ sector is given by

$$\begin{aligned} Z^-_{\text{vec}} = & \frac{1}{2\sinh \frac{\pm\epsilon_- + \epsilon_+}{2} \prod_{i=1}^{N+1} 2\cosh \frac{\pm\alpha_i + \epsilon_+}{2}} \prod_{I=1}^{n} \frac{2\cosh \frac{\pm\phi_I}{2} 2\cosh \frac{\pm\phi_I + 2\epsilon_+}{2}}{2\cosh \frac{\pm\phi_I \pm\epsilon_- + \epsilon_+}{2}} \\ & \times \prod_{I=1}^{n} \frac{2\sinh \epsilon_+}{2\sinh \frac{\pm\epsilon_- + \epsilon_+}{2} \prod_{i=1}^{N+1} 2\sinh \frac{\pm\phi_I \pm\alpha_i + \epsilon_+}{2}} \\ & \times \frac{\prod_{I>J}^{n} 2\sinh \frac{\pm\phi_I \pm\phi_J}{2} 2\sinh \frac{\pm\phi_I \pm\phi_J + 2\epsilon_+}{2}}{\prod_{I=1}^{n} 2\sinh \frac{\pm 2\phi_I \pm\epsilon_- + \epsilon_+}{2} \prod_{I>J}^{n} 2\sinh \frac{\pm\phi_I \pm\phi_J \pm\epsilon_- + \epsilon_+}{2}} \,, \end{aligned} \tag{3.49}$$

for $\chi = 1$ and

$$Z^-_{\text{vec}} = \frac{2\cosh\epsilon_+}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2} 2\sinh(\pm\epsilon_- + \epsilon_+)\prod_{i=1}^{N+1} 2\sinh(\pm\alpha_i + \epsilon_+)}$$
$$\times \prod_{I=1}^{n-1} \frac{2\sinh(\pm\phi_I) 2\sinh(\pm\phi_I + 2\epsilon_+)}{2\sinh(\pm\phi_I \pm \epsilon_- + \epsilon_+)} \prod_{I=1}^{n} \frac{2\sinh\epsilon_+}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}\prod_{i=1}^{N+1} 2\sinh\frac{\pm\phi_I\pm\alpha_i+\epsilon_+}{2}}$$
$$\times \frac{\prod_{I>J}^{n} 2\sinh\frac{\pm\phi_I\pm\phi_J}{2} 2\sinh\frac{\pm\phi_I\pm\phi_J+2\epsilon_+}{2}}{\prod_{I=1}^{n} 2\sinh\frac{\pm 2\phi_I\pm\epsilon_-+\epsilon_+}{2}\prod_{I>J}^{n} 2\sinh\frac{\pm\phi_I\pm\phi_J\pm\epsilon_-+\epsilon_+}{2}}\,, \tag{3.50}$$

for $\chi = 0$.

Here and below, repeated $\pm$ signs in the argument of the sinh functions mean multiplying all such functions. For instance,

$$2\sinh(\pm a \pm b + c)$$
$$\equiv 2\sinh(a + b + c) 2\sinh(a - b + c) 2\sinh(-a + b + c) 2\sinh(-a - b + c)\,. \tag{3.51}$$

Fundamental hypermultiplet index contribution for $O(k)^+$ sector is given by

$$Z^+_{\text{fund}} = \left(2\sinh\frac{m_l}{2}\right)^{\chi} \prod_{I=1}^{n} 2\sinh\frac{\pm\phi_I + m_l}{2}\,, \tag{3.52}$$

and for $O(k)^-$ sector is given by

$$Z^-_{\text{fund}} = 2\cosh\frac{m_l}{2} \prod_{I=1}^{n} 2\sinh\frac{\pm\phi_I + m_l}{2}\,, \tag{3.53}$$

for $\chi = 1$, and

$$Z^-_{\text{fund}} = 2\sinh\frac{m_l}{2} \prod_{I=1}^{n-1} 2\sinh\frac{\pm\phi_I + m_l}{2}\,, \tag{3.54}$$

for $\chi = 0$.

Because I am interested in the duality between 5d $Sp(2)$ and $SU(3)$ gauge theories with $N_f = 10$, let's focus on the $N = 1$ case of the index

One-instanton

One can see that there is no contour integral for one-instanton sector. The Witten index of the ADHM quantum mechanical system is given by the sum of $Z_1^{\pm}$

$$
\begin{aligned}
Z_{\text{ADHM}}^{k=1} &= \frac{1}{2}\left(Z_{\text{vec}}^{+}Z_{\text{fund}}^{+} + Z_{\text{vec}}^{-}Z_{\text{fund}}^{-}\right) \\
&= \frac{1}{2}\left(\frac{\prod_{l=1}^{10} 2\sinh\frac{m_l}{2}}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}\prod_{i=1}^{2} 2\sinh\frac{\pm\alpha_i+\epsilon_+}{2}} + \frac{\prod_{l=1}^{10} 2\cosh\frac{m_l}{2}}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}\prod_{i=1}^{2} 2\cosh\frac{\pm\alpha_i+\epsilon_+}{2}}\right) \\
&= \frac{t}{(1-tu)(1-t/u)}\frac{t^2}{(1-t^2v_1^2)(1-t^2/v_1^2)}\frac{t^2}{(1-t^2v_2^2)(1-t^2/v_2^2)} \\
&\quad\times\Bigg[-\left((v_1+\frac{1}{v_1})(v_2+\frac{1}{v_2})+(t+\frac{1}{t})^2\right)\chi_{512}^{SO(20)}(y_l) \\
&\qquad+\left(v_1+\frac{1}{v_1}+v_2+\frac{1}{v_2}\right)(t+\frac{1}{t})\chi_{\overline{512}}^{SO(20)}(y_l)\Bigg],
\end{aligned}
\tag{3.55}
$$

χ_R^G denotes character of G with representation R, so 512 and $\overline{512}$ are spinor and conjugate spinor representations of the group $SO(20)$. It shows manifest $SO(20)$ global symmetry. As I mentioned earlier, one-instanton sector doesn't contain any extra degrees, so this ADHM index is the one-instanton partition function

$$
Z_{\text{ADHM}}^{k=1} = Z_{\text{inst}}^{k=1}\,. \tag{3.56}
$$

In the next subsection, I will show the cases which contain extra degrees .

Perturbative index

One must include the perturbative partition function to obtain the BPS index. Perturbative index only includes W-bosons and matters contribution, and it is given by

$$
\begin{aligned}
Z_{\text{pert}} &= \text{PE}\left[\frac{t}{(1-tu)(1-t/u)}\left(-(t+\frac{1}{t})\chi^{Sp(2)}_{\text{adj},+}+\chi^{Sp(2)}_{\text{fund},+}\chi^{SO(20)}_{\text{fund}}\right)\right] \\
&= \text{PE}\Big[\frac{t}{(1-tu)(1-t/u)}\Big(-(t+\frac{1}{t})\left(v_1^2+v_2^2+v_1v_2+\frac{v_2}{v_1}\right) \\
&\qquad\qquad\qquad\qquad\qquad + (v_1+v_2)\,\chi^{SO(20)}_{\text{fund}}(y_l)\Big)\Big]\,,
\end{aligned} \tag{3.57}
$$

where $\chi^{Sp(2)}_{R,+}$ denotes the $Sp(2)$ character of the representation R, but only sums over positive weights. This is because the index acquires contribution only from quarks, W-bosons, and their superpartners, but not from anti-quarks or anti-W-bosons. I will use this notation throughout the paper. I chose the $Sp(2)$ positive roots by $2e_1, 2e_2, e_1+e_2$ and e_2-e_1 where e_1 and e_2 are orthogonal unit vectors. Plethystic exponential of $f(x)$ is defined by

$$
\text{PE}[f(x)] \equiv \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f(x^n)\right)\,, \tag{3.58}
$$

where x collectively denotes all the fugacities.

Full partition function

After collecting all ingredients, the full Nekrasov partition function of the 5d $Sp(2)$ gauge theory with $N_f=10$ is given by

$$
Z^{\text{5d},Sp(2)}_{\text{Nek}} = Z_{\text{pert}}\left(1+qZ^{k=1}_{\text{inst}}+\mathcal{O}(q^2)\right)\,, \tag{3.59}
$$

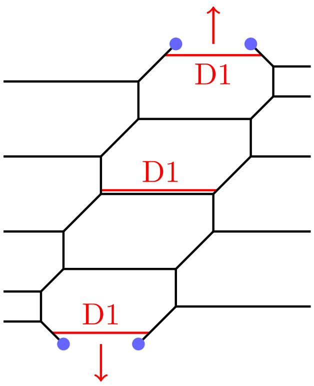
where all ingredients are given in (3.55), (3.57).

Unfortunately, I don't know how to extract two-instanton(or higher-order instantons) partition function from the Witten index of the $O(2)$ ADHM gauged quantum mechanics. Therefore, my result is limited to the one-instanton contribution. Note that the $Sp(2)$ index result can be easily generalized to the

$Sp(N+1)$ gauge group for arbitrary N. It will be used when I compare the 5d index with 6d $Sp(N)$ SCFT index. Now I want to compare this index with its dual $SU(3)$ gauge theory index.

3.1.3 Instanton partition function of $SU(3)$ gauge theory

I am interested in the $SU(3)$ gauge theory with 10 fundamental hypermultiplets. This theory can be realized by Type IIB brane system, which is drawn in Figure 3.3.

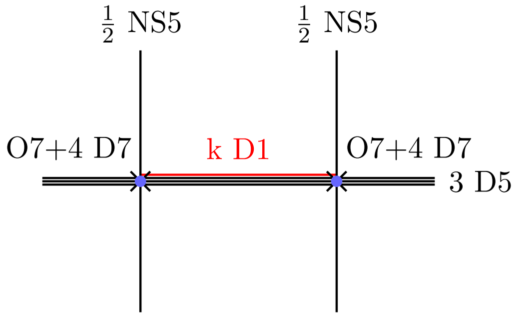


	0	1	2	3	4	5	6	7	8	9
D1	•	-	-	-	-	-	•	-	-	-
NS5	•	•	•	•	•	•	-	-	-	-
D5	•	•	•	•	•	-	•	-	-	-
D7	•	•	•	•	•	-	-	•	•	•

Figure 3.3 Type IIB brane diagram for the 5d $SU(3)$ gauge theory with 10 fundamental hypermultiplets. D1-branes engineer instanton soliton particles

Instantons are realized by D1-branes living on the D5-branes. Likewise $Sp(2)$ gauge theory with 10 fundamental hypermultiplets, this theory also contains uncontrollable extra degrees which are the escaping D1-branes to infinity. But the problem is worse than $Sp(2)$, because even one-instanton sector also has such unwanted degrees. In other words, it is unknown how to factor our the extra stringy contribution from ADHM Witten index. As a result, I can't compute 5d instanton partition function using the naive ADHM construction given in Figure 3.3.

To detour this problem, I use the group property of $SU(3)$, which is the $SU(3)$ '(anti)-fundamental' = 'anti-symmetric' representation. This is the special property for $SU(3)$ because there is an invariant tensor ϵ_{ijk}. Using such trick, I replace two fundamental hypermultiplets to two anti-symmetric hypermultiplets.[5] In this case, I have brane description, which is drawn in Figure 3.4. In this diagram, D1-brane can escape to infinity without feeling any force. Therefore, I can compute this contribution separately.



	0	1	2	3	4	5	6	7	8	9
D1	•	-	-	-	-	-	•	-	-	-
NS5	•	•	•	•	•	•	-	-	-	-
D5	•	•	•	•	•	-	•	-	-	-
O7-D7	•	•	•	•	•	-	-	•	•	•

Figure 3.4 Type IIB brane diagram for the 5d $SU(3)$ gauge theory with 8 fundamental and 2 anti-symmetric hypermultiplets. k D1-branes engineer k instanton solition particles

I want to compute k instanton partition function for the $SU(N)$ gauge group, and it can be obtained by considering dual 1d $U(k)$ AHDM gauge theory.

[5] Actually, this approach was already discussed in [52]. But they took the infinite mass limit of two anti-symmetric matters to obtain the index of 5d $SU(3)$ gauge theory with $N_f = 8$ fundamental matters. However, I will keep general masses of two anti-symmetric matters. Finally, I will see the global symmetry enhancement of $U(8)_f \times U(2)_a$ to $U(10)$, which is the evidence for my index is actually the $N_f = 10$ index.

Witten index for the ADHM gauged quantum mechanics is given by

$$Z^k_{\text{ADHM}} = \frac{(-1)^k}{k!} \oint \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} Z_{\text{vec}}(\phi, \alpha; \epsilon_{1,2}) \prod_{l=1}^{8} Z_{R_f}(\phi, \alpha, m_l; \epsilon_{1,2}) \prod_{a=1}^{2} Z_{R_a}(\phi, \alpha, z_a; \epsilon_{1,2}) \,, \tag{3.60}$$

where Z_{vec} is 5d vector multiplet contribution to the index, Z_{R_f} is the 5d fundamental hypermultiplet and Z_{R_a} is the 5d anti-symmetric hypermultiplet contribution to the index with mass m_f and z_a each [52, 64]. I will use the fugacity variables $v_i = e^{-\alpha_i}$, $y_l = e^{-m_l}, \tau_a = e^{-z_a}, t = e^{-\epsilon_+}, u =^{-\epsilon_-}$. The vector multiplet contribution to the index is given by

$$Z_{\text{vec}}(\phi, \alpha; \epsilon_{1,2}) = \frac{\prod_{I\neq J}^{k} 2\sinh\frac{\phi_I-\phi_J}{2} \prod_{I,J}^{k} 2\sinh\frac{\phi_I-\phi_J+2\epsilon_+}{2}}{\prod_{I,J}^{k} 2\sinh\frac{\phi_I-\phi_J+\epsilon_1}{2} 2\sinh\frac{\phi_I-\phi_J+\epsilon_2}{2} \prod_{i=1}^{N}\prod_{I=1}^{k} 2\sinh\frac{\pm(\phi_I-\alpha_i)+\epsilon_+}{2}} \,. \tag{3.61}$$

If one is interested in $SU(N)$ gauge theory, traceless condition $\sum_{i=1}^{N} \alpha_i = 0$ can be imposed. The fundamental hypermultiplet contribution to the index is given by

$$Z_{\text{fund}}(\phi, m_l; \epsilon_{1,2}) = \prod_{I=1}^{k} 2\sinh\frac{\phi_I - m_l}{2} \,, \tag{3.62}$$

and anti-symmetric hypermultiplet contribution to the index is given by

$$\begin{aligned} &Z_{\text{anti-sym}}(\phi, \alpha, z_a; \epsilon_{1,2}) \\ &= \frac{\prod_{i=1}^{N}\prod_{I=1}^{k} 2\sinh\frac{\phi_I+\alpha_i-z_a}{2} \prod_{I>J}^{k} 2\sinh\frac{\phi_I+\phi_J-z_a-\epsilon_-}{2} 2\sinh\frac{-\phi_I-\phi_J+z_a-\epsilon_-}{2}}{\prod_{I>J}^{k} 2\sinh\frac{\phi_I+\phi_J-z_a-\epsilon_+}{2} 2\sinh\frac{-\phi_I-\phi_J+z_a-\epsilon_+}{2} \prod_{I=1}^{k} 2\sinh\frac{2\phi_I-z_a-\epsilon_+}{2} 2\sinh\frac{-2\phi_I+z_a-\epsilon_+}{2}} \,. \end{aligned} \tag{3.63}$$

Now I am ready to compute instanton partition functions. For later convenience, I will find and use plethystic exponential form of ADHM partition function and extra contribution, i.e.

$$Z_{\text{ADHM}} = \text{PE}\Big[qF^{k=1}_{\text{ADHM}} + q^2 F^{k=2}_{\text{ADHM}} + \cdots\Big]\,, \tag{3.64}$$

$$Z_{\text{extra}} = \text{PE}\Big[qF^{k=1}_{\text{extra}} + q^2 F^{k=2}_{\text{extra}} + \cdots\Big]\,. \tag{3.65}$$

For one-instanton sector, $F^{k=1}_{\text{ADHM}} = Z^{k=1}_{\text{ADHM}}$ and $F^{k=1}_{\text{extra}} = Z^{k=1}_{\text{extra}}$. For two-instanton sector,

$$F^{k=2}_{\text{ADHM}} = Z^{k=2}_{\text{ADHM}} - \frac{1}{2}\left((Z^{k=1}_{\text{ADHM}})^2 + Z^{k=1}_{\text{ADHM}}(* \to *^2)\right)\,, \tag{3.66}$$

where $(*)$ collectively denotes all the fugacities.

One-Instanton

Let's first consider one-instanton partition function, i.e. $k=1$. The Witten index for $U(1)$ ADHM gauged QM is given by the contour integral, which is given by

$$\begin{aligned} &Z^{k=1}_{\text{ADHM}} \\ &= -\oint \frac{d\phi_1}{2\pi i}\left[\frac{2\sinh\epsilon_+}{2\sinh\frac{\epsilon_1}{2}2\sinh\frac{\epsilon_2}{2}}\frac{\left(\prod_{l=1}^{8} 2\sinh\frac{\phi_1 - m_l}{2}\right)}{\prod_{i=1}^{3} 2\sinh\frac{\phi_1-\alpha_i+\epsilon_+}{2} 2\sinh\frac{-\phi_1+\alpha_i+\epsilon_+}{2}}\right. \\ &\qquad\left.\times \prod_{a=1}^{2}\frac{\prod_{i=1}^{3}\sinh(\frac{\phi_1+\alpha_i-z_a}{2})}{2\sinh\frac{2\phi_1 - z_a - \epsilon_+}{2}2\sinh\frac{-2\phi_1+z_a-\epsilon_+}{2}}\right]\,. \end{aligned} \tag{3.67}$$

The contour choice is given by JK-residue (3.42). If one takes the reference vector to $\mathfrak{n}=1$, the poles that give non-trivial JK residue are given by

$$\phi_1 - \alpha_i + \epsilon_+ = 0\,, \quad 2\phi_1 - z_a - \epsilon_+ = 0 \ (\text{mod}\ 2\pi)\,. \tag{3.68}$$

Therefore, there are seven poles $\phi_* = \alpha_{1,2,3} - \epsilon_+$, $\phi_* = \frac{1}{2}(z_{1,2} - \epsilon_+)$, $\phi_* = \frac{1}{2}(z_{1,2} - \epsilon_+ + 2\pi i)$. After computing all JK-residues at seven poles ϕ_* and collecting all of them, one can obtain the Witten index for $U(1)$ ADHM gauged

QM $Z^{k=1}_{\text{ADHM}}$. To obtain one-instanton partition function from this, one has to consider extra freely escaping D1-brane degree, and then one should extract the Z_{extra} from Z_{ADHM}. Such Z_{extra} is given by Coulomb VEV parameters α_i independent parts in Z_{AHDM}, and the extra contribution at one-instanton order is conjectured by

$$\begin{aligned}
&F^{k=1}_{\text{extra}}(y,\tau;t,u)\left(=Z^{k=1}_{\text{extra}}(y,\tau;t,u)\right)\\
&=\frac{-\sqrt{\tau_1\tau_2}}{(1-tu)(1-t/u)(1-t^2\tau_1/\tau_2)(1-t^2\tau_2/\tau_1)}\\
&\times\left(t^2(1+t^2)\left(\chi^{U(8)}_{\Lambda^6}+\frac{1}{\tau_1\tau_2}\chi^{U(8)}_{\Lambda^2}\right)+t^3(\tau_1+\tau_2)\left(1+\frac{1}{\tau_1\tau_2}\chi^{U(8)}_{\Lambda^4}+\frac{1}{\tau_1^2\tau_2^2}\chi^{U(8)}_{\Lambda^0}\right)\right)\\
&+\frac{1}{(1-tu)(1-t/u)}\left(\frac{1}{\sqrt{\tau_1\tau_2}}\chi^{U(8)}_{\Lambda^8}+t^2\sqrt{\tau_1\tau_2}\chi^{U(8)}_{\Lambda^0}\right)\,,
\end{aligned}\tag{3.69}$$

where $\chi^{U(8)}_{\Lambda^L}$ is the normalized character of rank L anti-symmetric irreducible representation of $U(N_f=8)$ flavor group with fugacities y_l

$$\chi^{U(8)}_{\Lambda^L}\equiv\left(\frac{1}{\prod_{l=1}^8 y_l}\right)^{\frac{1}{2}}\sum_{1\le l_1\le l_2\le\cdots\le l_L\le 8}y_{l_1}y_{l_2}\cdots y_{l_L}\,,\tag{3.70}$$

with $\chi^{U(8)}_{\Lambda^0}=\left(\frac{1}{\prod_{l=1}^8 y_l}\right)^{\frac{1}{2}}=\left(\chi^{U(8)}_{\Lambda^8}\right)^{-1}$ This extra factor (3.69) is already obtained in [52], and they conjectured that there is no two-instanton contribution to the extra factor(i.e. $F^{k=2}_{\text{extra}}=0$). But I will show that actually there is two-instanton correction.

After subtracting extra contribution from ADHM quantum mechanics index, one can obtain exact one-instanton partition function for 5d $SU(3)$ gauge theory with $N_f=8$ and $N_a=2$ hypermultiplets

$$F^{k=1}_{\text{inst}}=F^{k=1}_{\text{ADHM}}-F^{k=1}_{\text{extra}}\,.\tag{3.71}$$

Its form is very complicated to write here. I first show the t-expanded form to check the global symmetry enhancement. Using the $\chi^{U(8)}_{\Lambda^L}(y_l)$ notation, the t-expanded index is given by

$$F^{k=1}_{\text{inst}} = t^2 \frac{1}{\sqrt{\tau_1\tau_2}} \Bigg(\chi^{U(8)}_{\Lambda^0} + \chi^{U(8)}_{\Lambda^6} + \chi^{SU(3)}_{3}(v_i)\chi^{U(8)}_{\Lambda^8} + (\tau_1+\tau_2)\chi^{U(8)}_{\Lambda^1} + (\tau_1+\tau_2)\chi^{U(8)}_{\Lambda^7} \\ + \tau_1\tau_2\chi^{U(8)}_{\Lambda^2} + \tau_1\tau_2\chi^{U(8)}_{\Lambda^8} + \tau_1\tau_2\chi^{SU(3)}_{\bar{3}}(v_i) \Bigg) + t^4(\cdots) + \cdots\,, \qquad (3.72)$$

where $\chi^{SU(3)}_{R}(v_i)$ is the $SU(3)$ group character of the representation R with fugacities v_i. For example, $\chi^{SU(3)}_{3}(v_i) = v_1 + v_2 + \frac{1}{v_1 v_2}$.

If one take $\tau_1 = 1/y_9$ and $\tau_2 = 1/y_{10}$, above index (3.72) can be rewritten with $\chi^{U(10)}_{\Lambda^L}$ which is the normalized character of rank L anti-symmetric irreducible representation of $U(N_f = 10)$ with fugacities y_l, i.e.

$$F^{k=1}_{\text{inst}} = t^2 \left(\chi^{U(10)}_{\Lambda^2} + \chi^{U(10)}_{\Lambda^8} + \chi^{SU(3)}_{3}(v_i)\chi^{U(10)}_{\Lambda^{10}} + \chi^{SU(3)}_{\bar{3}}(v_i)\chi^{U(10)}_{\Lambda^0} \right) + t^4(\cdots) + \cdots\,. \qquad (3.73)$$

Therefore, this instanton partition function truly shows the $U(10)$ flavor symmetry enhancement of the $U(8)\times U(2)$ flavor symmetry, and it is expected property of $N_f = 10$ theory. After taking $\tau_1 = 1/y_9$ and $\tau_2 = 1/y_{10}$ in ordinary instanton partition function, one can obtain following explicit form of one-instanton partition function

$$F^{k=1}_{\text{inst}} = F^{k=1}_{\text{naive}} + F^{k=1}_{\text{add}}\,, \qquad (3.74)$$

where $F^{k=1}_{\text{naive}}$ and $F^{k=1}_{\text{add}}$ are given by

$$F^{k=1}_{\text{naive}} = \frac{-t}{(1-tu)(1-t/u)} \left[\frac{t^3 \sum_{L=0}^{10} \chi^{U(10)}_{\Lambda^L} (-\frac{v_1}{t})^{6-L}}{(v_1-v_2)(v_1-t^2 v_2)(v_1-v_3)(v_1-t^2 v_3)} + (\text{cyclic}) \right]\,, \qquad (3.75)$$

$$
\begin{aligned}
F_{\rm add}^{k=1} = & \frac{-t}{(1-tu)(1-t/u)} \\
& \Big[-\frac{1}{t^3}\chi_{\Lambda^0}^{U(10)}\left(\chi_3^{SU(3)}(v_i^2)+(1+t^2)\chi_{\bar{3}}^{SU(3)}(v_i)\right)+\frac{1}{t^2}\chi_{\Lambda^1}^{U(10)}\chi_3^{SU(3)}(v_i)-\frac{1}{t}\chi_{\Lambda^2}^{U(10)} \\
& -t^3\chi_{\Lambda^{10}}^{U(10)}\left(\chi_{\bar{3}}^{SU(3)}(v_i^2)+(1+\frac{1}{t^2})\chi_3^{SU(3)}(v_i)\right)+t^2\chi_{\Lambda^9}^{U(10)}\chi_{\bar{3}}^{SU(3)}(v_i)-t\chi_{\Lambda^8}^{U(10)}\Big] ,
\end{aligned}
\tag{3.76}
$$

where $SU(3)$ condition has to be imposed by $v_1v_2v_3 = 1$. This result agrees with un-refined topological vertex computation($t = 0$ limit) [57].

Two-instantons

Let's consider two-instanton partition function, i.e. $k = 2$. The Witten index for $U(2)$ ADHM gauged QM is given by the contour integral of two complex variables ϕ_1, ϕ_2, which is given by

$$
\begin{aligned}
& Z_{\rm ADHM}^{k=2} \\
& = \frac{1}{2}\oint \frac{d\phi_1}{2\pi i}\frac{d\phi_2}{2\pi i}\Bigg[\Bigg(\left(\frac{2\sinh\epsilon_+}{2\sinh\frac{\epsilon_1}{2}2\sinh\frac{\epsilon_2}{2}}\right)^2 \frac{2\sinh\frac{\pm(\phi_1-\phi_2)}{2}2\sinh\frac{\pm(\phi_1-\phi_2)+\epsilon_+}{2}}{2\sinh\frac{\pm(\phi_1-\phi_2)+\epsilon_1}{2}2\sinh\frac{\pm(\phi_1-\phi_2)+\epsilon_2}{2}} \\
& \times \frac{1}{\prod_{i=1}^N\prod_{I=1}^2 2\sinh\frac{\pm(\phi_I-\alpha_i)+\epsilon_+}{2}}\Bigg)\left(\prod_{l=1}^8\prod_{I=1}^2 2\sinh\frac{\phi_I-m_l}{2}\right) \\
& \times \prod_{a=1}^2\left(\frac{\prod_{i=1}^3\prod_{I=1}^2 2\sinh\frac{\phi_I-\alpha_i-z_a}{2}2\sinh\frac{\pm(\phi_1+\phi_2-z_a)-\epsilon_-}{2}}{2\sinh\frac{\pm(\phi_1+\phi_2-z_a)-\epsilon_+}{2}\prod_{I=1}^2 2\sinh\frac{\pm(2\phi_I-z_a)-\epsilon_+}{2}}\right)\Bigg] .
\end{aligned}
\tag{3.77}
$$

The contour choice is given by JK-residue (3.42). If the reference vector is taken to $\mathfrak{n} = (1, 1)$,[6] one can obtain non-trivial JK residue at the poles $\phi_* = (\phi_1, \phi_2)$.

[6]For actual computation, I chose $\mathfrak{n} = (1, 1.01)$.

They are classified as follows

$$
\begin{aligned}
&1)\quad \phi_* = (\alpha_i - \epsilon_+, \alpha_j - \epsilon_+) \quad \text{where} \quad i \neq j\,,\\
&2)\quad \phi_* = (\alpha_i - \epsilon_+, \alpha_i - \epsilon_+ - \epsilon_{1,2}) \quad \text{and} \quad \phi_* = (\alpha_i - \epsilon_+ - \epsilon_{1,2}, \alpha_i - \epsilon_+)\,,\\
&3)\quad \phi_* = (-\alpha_i + 2\epsilon_+ + z_a, \alpha_i - \epsilon_+)\,,\\
&4)\quad \phi_* = (\frac{\epsilon_+ + z_a}{2}, \alpha_i - \epsilon_+) \quad \text{and} \quad \phi_* = (\frac{\epsilon_+ + z_a}{2} + i\pi, \alpha_i - \epsilon_+)\,,\\
&\qquad \phi_* = (\alpha_i - \epsilon_+, \frac{\epsilon_+ + z_a}{2}) \quad \text{and} \quad \phi_* = (\alpha_i - \epsilon_+, \frac{\epsilon_+ + z_a}{2} + i\pi)\,,\\
&5)\quad \phi_* = (\frac{\epsilon_+ + z_a + \epsilon_{1,2}}{2}, \frac{\epsilon_+ + z_a - \epsilon_{1,2}}{2})\\
&\qquad \text{and} \quad \phi_* = (\frac{\epsilon_+ + z_a + \epsilon_{1,2}}{2} + i\pi, \frac{\epsilon_+ + z_a - \epsilon_{1,2}}{2} + i\pi)\\
&6)\quad \phi_* = (\frac{-\epsilon_+ + z_a}{2}, \frac{3\epsilon_+ - z_a}{2} + z_b)\\
&\qquad \text{and} \quad \phi_* = (\frac{-\epsilon_+ + z_a}{2} + i\pi, \frac{3\epsilon_+ - z_a}{2} + z_b + i\pi) \quad \text{where} \quad a \neq b\\
&7)\quad \phi_* = (\frac{\epsilon_+ - z_a}{2} + z_b, \frac{\epsilon_+ + z_a}{2})\\
&\qquad \text{and} \quad \phi_* = (\frac{\epsilon_+ - z_a}{2} + z_b + i\pi, \frac{\epsilon_+ + z_a}{2} + i\pi) \quad \text{where} \quad a \neq b\\
&8)\quad \phi_* = (\frac{\epsilon_+ + z_a}{2}, \frac{\epsilon_+ + z_b}{2}) \quad \text{and} \quad \phi_* = (\frac{\epsilon_+ + z_a}{2} + i\pi, \frac{\epsilon_+ + z_b}{2})\\
&\qquad \text{and} \quad \phi_* = (\frac{\epsilon_+ + z_a}{2}, \frac{\epsilon_+ + z_b}{2} + i\pi) \quad \text{and} \quad \phi_* = (\frac{\epsilon_+ + z_a}{2} + i\pi, \frac{\epsilon_+ + z_b}{2} + i\pi)\\
&\qquad \text{where} \quad a \neq b\\
&9)\quad \phi_* = (\frac{\epsilon_+ + z_a}{2} + i\pi, \frac{\epsilon_+ + z_a}{2}) \quad \text{and} \quad \phi_* = (\frac{\epsilon_+ + z_a}{2}, \frac{\epsilon_+ + z_a}{2} + i\pi)\,,
\end{aligned}
\tag{3.78}
$$

where $i, j = 1, 2, 3$ and $a, b = 1, 2$. There are 76 poles, so two-instanton partition function is given by the sum of 76 residues. Firstly, one needs to inspect extra stringy contribution to the index, which is given by Coulomb VEV $v_{i=1,2,3}$ independent part in $F^{k=2}_{\text{ADHM}}$ given in (3.66). In the work [52], they found that there is no two-instanton extra contribution to the index, i.e. $F^{k=2}_{\text{ADHM}} = 0$ until

$N_f < 6$. But I find there is two-instanton extra contribution to the index for $N_f = 8$ case, and they are given by

$$
\begin{aligned}
F^{k=2}_{\text{extra}} = & \frac{-t^3}{(1-tu)(1-t/u)(1-t^2\tau_1/\tau_2)(1-t^2\tau_2/\tau_1)} \Bigg[\left(u + \frac{1}{u} + t + \frac{1}{t}\right) \\
& + \frac{3}{2}\left((u+\frac{1}{u})(t+\frac{1}{t}) + \frac{1}{t^3} + \frac{1}{t} + t + t^3\right) - \left(\frac{\tau_1}{\tau_2} + \frac{\tau_2}{\tau_1}\right)\left(u + \frac{1}{u} + t + \frac{1}{t}\right) \\
& + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\sum_{i>j} y_i y_j + (\tau_1+\tau_2)\sum_{i>j}\frac{1}{y_i y_j} + \left(t + \frac{1}{t}\right)\sum_{i,j}\frac{y_i}{y_j}\Bigg] .
\end{aligned} \tag{3.79}
$$

It must be inserted in (3.65). After subtracting $F^{k=2}_{\text{extra}}$ factor from $F^{k=2}_{\text{ADHM}}$, one can obtain two-instanton partition function $F^{k=2}_{\text{inst}}$ correctly. Its form is rather complicated to write here. Its t-expanded form is given by

$$
\begin{aligned}
F^{k=2}_{\text{inst}} = & 1 + 2\,t\,\left(\frac{1}{u} + u\right) + \,t^2\,\left(2\left(u + \frac{1}{u}\right)^2 + 1 + \chi^{U(10)}_{\text{fund}}\chi^{U(10)}_{\text{anti-fund}}\right) \\
& + t^3\left(2\left(\frac{1}{u} + u\right)^3 + \chi^{U(10)}_{\Lambda^3} + \chi^{U(10)}_{\Lambda^7} + \left(u + \frac{1}{u}\right)\chi^{U(10)}_{\text{fund}}\chi^{U(10)}_{\text{anti-fund}}\right) + \dots ,
\end{aligned} \tag{3.80}
$$

where $\chi^{U(10)}_{\text{fund}} = \sum_{l=1}^{10} y_l$ and $\chi^{U(10)}_{\text{anti-fund}} = \sum_{l=1}^{10}\frac{1}{y_l}$. Again, one can see the $U(10)$ flavor symmetry enhancement. Exact result of two-instanton partition function also agrees with un-refined topological vertex computation($t = 0$ limit) [57]. Because of computational complexity, my results are limited up to two-instanton partition function. But one can study higher-order instanton partition function with index formula (3.60) and JK-residue prescription (3.42).

Perturbative index

Likewise $Sp(2)$ case, one must include the perturbative partition function to obtain full index. Perturbative index only includes W-bosons and matters contribution, and it is given by

$$Z_{\text{pert}} = \text{PE}[F_{\text{pert}}]\,, \tag{3.81}$$

where F_{pert} is given by

$$\begin{aligned}
F_{\text{pert}} &= \text{PE}\left[\frac{t}{(1-tu)(1-t/u)}\left(-(t+\frac{1}{t})\chi^{SU(3)}_{\text{adj},+} + \chi^{SU(3)}_{\text{fund},+}\chi^{U(10)}_{\text{ant-ifund}} + \chi^{SU(3)}_{\text{anti-fund},+}\chi^{U(10)}_{\text{fund}}\right)\right] \\
&= \text{PE}\Big[\frac{t}{(1-tu)(1-t/u)}\Big(-(t+\frac{1}{t})\left(v_1^2 v_2 + \frac{v_1}{v_2} + v_1 v_2^2\right) \\
&\qquad\qquad + (v_1+v_2)\sum_{l=1}^{10}\frac{1}{y_l} + v_1 v_2 \sum_{l=1}^{10} y_l\Big)\Big]\,.
\end{aligned} \tag{3.82}$$

Full index

After collecting all ingredients, one can obtain Nekrasov partition function of the 5d $SU(3)$ gauge theory with $N_f = 10$ fundamental matters

$$Z^{\text{5d},SU(3)}_{\text{Nek}} = \text{PE}[q^0 F_{\text{pert}} + q^1 F^{k=1}_{\text{inst}} + q^2 F^{k=2}_{\text{inst}} + \mathcal{O}(q^3)]\,, \tag{3.83}$$

where all ingredients are given in (3.74), (3.80), (3.82).

3.1.4 Duality test between 5d gauge theories

I obtained Nekrasov partition functions of the conjectured two dual theories. To test the duality, I need to compare their indices and confirm that they are same. But at first glance they do not match. To see that they are really same, one needs non-trivial fugacity map between two indices.

Fugacity map

Fortunately, fugacity map between two theories was obtained in the work [57]. They obtained the fugacity map by comparing the Type IIB (p, q)-web diagrams of two theories. If one use unprimed variables as 5d $SU(3)$ fugacities (q, v_i, y_l) and primed variables as $Sp(2)$ fugacities (q', v'_i, y'_l), then fugacity map is given as

$$\begin{aligned}
\text{Instanton} \ &: \ q' = q \ , \\
\text{Gauge} \ &: \ v'_i = \left(q^{\frac{1}{2}} \prod_{j=1}^{10} y_j^{-\frac{1}{4}} \right) v_i \ , \\
\text{Flavor} \ &: \ y'_l = \left(q^{\frac{1}{2}} \prod_{j=1}^{10} y_j^{-\frac{1}{4}} \right) y_l \quad \text{for} \quad l = 1, \ldots, 5 \ , \\
& \quad\ y_l'^{-1} = \left(q^{\frac{1}{2}} \prod_{j=1}^{10} y_j^{-\frac{1}{4}} \right) y_l \quad \text{for} \quad l = 6, \ldots, 10 \, .
\end{aligned}$$

Instanton fugacity is mapped instanton fugacity itself, but other fugacities receive a non-trivial correction and its origin is well-explained in [57]. Under the above fugacity map, I check the agreement of two indices. This shows the $SO(20)$ global symmetry enhancement in UV for the 5d $SU(3)$ gauge theory with $N_f = 10$ whose global symmetry seems $U(10)$ in low energy. This is the evidence of the duality of two 5d gauge theories.

3.2 6d $\mathcal{N} = (1, 0)$ QFTs

In this section, I will discuss about 6d $Sp(N)$ SCFT with $2N + 8$ fundamental matters and their supersymmetric indices. The case with $N = 0$ is the famous E-string theory. 6d is the highest dimensions that superconformal field theories can exist. Recently, possible 6d $\mathcal{N} = (1, 0)$ SCFTs are classified by F-theory compactification on elliptic Calabi-Yau 3-folds [45, 85, 86]. Basic field content of 6d theories are tensor multiplet, which consists of a self-dual tensor, a real scalar, and their superpartners. One can add gauge symmetry and couple matter fields. Gauge and global anomaly conditions restrict possible gauge groups and matter contents. For instance, $SU(2)$ gauge group with $N_f = 4, 10$ fundamental matters, $SU(3)$ gauge group with $N_f = 0, 6, 12$ fundamental matters, gauge group G_2 with $N_7 = 1, 4, 7$ fundamental matters are possible. Gauge group with F_4, E_6, E_7, E_8 are also possible. I'm interested in the 6d SCFT with $Sp(N)$ gauge group and $N_f = 2N + 8$ fundamental matters, which can be higgsed to the $SU(2)$ with $N_f = 10$ theory and finally E-string theory [87].

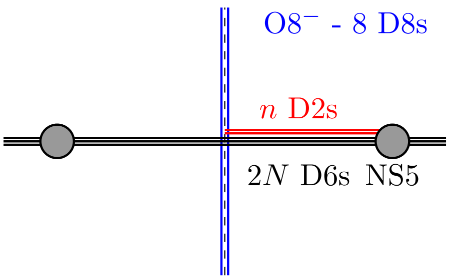
There are tensionless self-dual strings in 6d SCFT which couple to the self-dual tensor field. These self-dual strings have tensions in the tensor branch where tensor multiplet scalar has non-zero VEV. The tension of self-dual strings is proportional to the tensor multiplet scalar VEV. On the tensor branch, 6d SCFT have 6d effective super-Yang-Mills descriptions, and inverse gauge coupling is proportional the the tension of self-dual string. In 6d effective super-Yang-Mills descriptions, self-dual strings are realized by self-dual Yang-Mills instanton soliton strings. Therefore, in the strong coupling limit, instanton soliton strings become massless and they are playing an important role in the CFT limit. I will study the 2d $\mathcal{N} = (0, 4)$ gauge theory description of such instanton soliton strings and compute their elliptic genus.

3.2.1 6d $Sp(1)$ SCFT with 10 fundamental matters

I will study the circle compactified 6d SCFT with $Sp(1)$ gauge symmetry and its 5d $Sp(2)$ gauge theory description. Both theories have $N_f = 10$ fundamental hypermultiplets. I want to confirm the duality by comparing the 5d instanton partition function and the elliptic genera of the self-dual strings in the 6d theory.

The elliptic genera of the 6d $Sp(1)$ gauge theory are partially studied in [87]. Main difference between [87] and my computation is the presence of the E_8(or SO(20)) Wilson line. The 6d theory can be Higgs to the E-string theory, and circle compactified E-string theory has the effective 5d gauge theory description with $Sp(1)$ gauge group and $N_f = 8$ fundamental matters. So for duality to hold, one has to turn on the background $SO(20)$ Wilson line which reduces to the E_8 Wilson line after higgsing. I will show that the 5d and 6d indices agree with each other after this shift. $SO(20)$ Wilson line effect will be discussed in more details later.

First consider type IIA brane description of the 6d $\mathcal{N} = (1,0)$ SCFT with $Sp(N)$ gauge symmetry and $N_f = 2N+8$ hypermultiplets. The case with $N = 0$ engineers the E-string theory and $N = 1$ is conjectured to the dual of above two 5d gauge theories. Brane system is given in Figure 3.5.



	0	1	2	3	4	5	6	7	8	9
D2	•	•	-	-	-	-	•	-	-	-
NS5	•	•	•	•	•	•	-	-	-	-
D6	•	•	•	•	•	•	•	-	-	-
$O8^-$-D8	•	•	•	•	•	•	-	•	•	•

Figure 3.5 Type IIA brane system for 6d $\mathcal{N} = (1,0)$ $Sp(N)$ gauge theory with $N_f = 2N + 8$ fundamental hypermultiplets. n D2-branes engineer n self-dual strings.

I focus on the self dual-strings which couple to the tensor multiplet in the 6d SCFT. The self-dual strings are instanton soliton strings in 6d gauge theory, and it is realized by D2-branes living on D6-branes. The quiver diagram for the 2d $\mathcal{N} = (0, 4)$ gauge theory living on D2-branes is given in Figure 3.6. Their SUSY and Lagrangian are studied in [87, 88]. $O(n)$ vector multiplet and

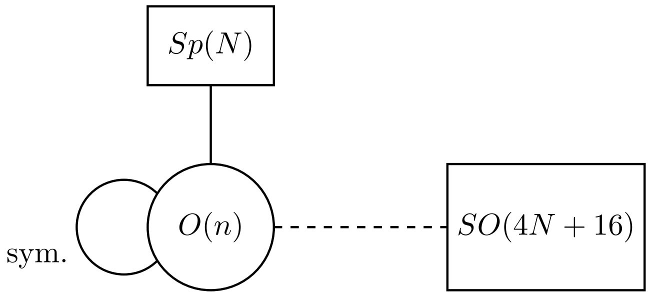


Figure 3.6 2d ADHM quiver diagram for the self-dual strings.

symmetric hypermultiplet come from the strings stretch between D2-D2 branes with appropriate boundary conditions in the presence of O8$^-$-plane. Hypermultiplets whose representation is $(n, 2N)$ come from D2-D6 strings, and Fermi multiplets whose representation is $(n, 4N + 16)$ come from D2-D8 strings and D2-D6 strings across NS5 brane. I circle compactify the theory along x^1 direction.

3.2.2 Elliptic genera of self-dual strings

The definition of BPS index for the 6d $Sp(1)$ theory with $N_f = 10$ is similar with 5d index. It consists of perturbative part and non-perturbative part $Z^{\text{6d},Sp(N)} = Z^{\text{6d}}_{\text{pert}} Z^{\text{6d}}_{\text{s.d.}}$, and later one can be computed by elliptic genera of the self-dual strings. I focus on elliptic genera of the self-dual strings of the 6d $Sp(N)$ theories

$$Z^{\text{6d}}_{\text{s.d.}} = 1 + \sum_{n=1}^{\infty} w^n Z^n_{\text{s.d.}} \,, \tag{3.84}$$

where w is the fugacity for the string winding number. The elliptic genus of the 2d gauge theory on a tours is

$$Z^n_{\text{s.d.}} = \text{Tr}_{RR}\left[(-1)^F q^{2H_L}\bar{q}^{2H_R} e^{2\pi i\epsilon_1(J_1+J_R)} e^{2\pi i\epsilon_2(J_2+J_R)} \prod_{i=1}^{N} e^{2\pi i\alpha_i G_i} \prod_{l=1}^{N_f=2N+8} e^{2\pi i m_l F_l}\right]. \tag{3.85}$$

$q \equiv e^{i\pi\tau}$ contains the complex structure of the torus τ.[7] $H_R \sim \{Q, Q^\dagger\}$ where $Q, Q^\dagger$ are $(0,2)$ supercharges of the theory. J_1, J_2 and J_R are Cartans of $SO(4)_{2345}$ and $SO(3)_{789} \sim SU(2)_R$. G_i are Cartans of $Sp(N)$ gauge group of 6d SCFT and α_i are corresponding chemical potentials. F_l are Cartans of $SO(4N+16)$ flavor symmetry and m_l are corresponding chemical potentials. The elliptic genus of n E-strings is given by $N = 0$ case. The elliptic genus of the 2d gauge theory (3.85) was studied in [81–83], and the E-string case(or $O(n)$ gauge group) was further studied in [87, 88]. The elliptic genus is given by an integral over the $O(n)$ flat connections on T^2. $O(n)$ gauge group has two disconnected parts $O(n)^\pm$. So the Wilson lines U_1, U_2 along the temporal and spatial circle have two disconnected sector. The discrete holonomy sectors for $O(n)$ gauge group on T^2 are listed in section 3 of [88]. Usually elliptic genus is given by sum of 8 discrete sectors for a given n. But $n = 1$ and $n = 2$ cases are special, and they are given by sum of 4 and 7 sectors respectively.

The elliptic genus (3.85) is given by [81, 88]

$$Z_n^{6\text{d},Sp(N)} = \sum_I \frac{1}{|W_I|}\frac{1}{(2\pi i)^r}\oint Z^{(I)}_{\text{1-loop}}\ , \quad Z^{(I)}_{\text{1-loop}} \equiv Z^{(I)}_{\text{vec}} Z^{(I)}_{\text{sym.}} Z^{(I)}_{\text{Fermi}} Z^{(I)}_{\text{fund.}}\ . \tag{3.86}$$

[7] I use definition of q as $q \equiv e^{i\pi\tau}$ instead of usual $q \equiv e^{2i\pi\tau}$, because instanton fugacity in 5d gauge theory correspond with this definition of q.

The 1-loop determinant for the 2d multiplets are given by

$$Z_{\text{vec}} = \prod_{i=1}^{r}\left(\frac{2\pi\eta^2 du_i}{i}\cdot\frac{\theta_1(2\epsilon_+)}{i\eta}\right)\prod_{\alpha\in\text{root}}\frac{\theta_1(\alpha(u))\theta_1(2\epsilon_++\alpha(u))}{i\eta^2}\,, \tag{3.87}$$

$$Z_{\text{sym hyper}} = \prod_{\rho\in\text{sym}}\frac{i\eta}{\theta_1(\epsilon_1+\rho(u))}\frac{i\eta}{\theta_1(\epsilon_2+\rho(u))}\,, \tag{3.88}$$

$$Z_{\text{Fermi}}^{SO(4N+16)} = \prod_{\rho\in\text{fund}}\prod_{l=1}^{2N+8}\frac{\theta_1(m_l+\rho(u))}{i\eta}\,, \tag{3.89}$$

$$Z_{\text{fund hyper}}^{Sp(N)} = \prod_{\rho\in\text{fund}}\prod_{i=1}^{N}\frac{i\eta}{\theta_1(\epsilon_++\rho(u)+\alpha_i)}\frac{i\eta}{\theta_1(\epsilon_++\rho(u)-\alpha_i)}\,, \tag{3.90}$$

where $\epsilon_\pm \equiv \frac{\epsilon_1\pm\epsilon_2}{2}$ and r is the rank of the dual gauge group $O(n)$. $\eta \equiv \eta(\tau)$ is the Dedekind eta function and $\theta_i(z) \equiv \theta_i(\tau, z)$ are the Jacobi theta functions. 'I' refers the disconnected holonomy sectors and u_i are zero modes of 2d gauge fields along the torus. $|W_I|$ is order of Weyl group of $O(n)_I$ for each sector 'I' [88]. For later convenience, I will use following fugacity notation $t \equiv e^{2\pi i\epsilon_+}$, $u \equiv e^{2\pi i\epsilon_-}$, $v_i \equiv e^{2\pi i\alpha_i}$, $y_l \equiv e^{2\pi i m_l}$. The elliptic genus contains contour integral of u_i, which is a residue sum given by Jeffrey-Kirwan residue(JK-residue) prescription (3.42).

$SO(20)$ Wilson line effect

The E-string elliptic genus has manifest E_8 global symmetry. One should turn on the E_8 Wilson line on a circle to obtain 5d SYM description of E-string theory [88].[8] This background E_8 Wilson line provides the extra shift $m_8 \to m_8 - \tau$ to the chemical potential. So it gives following shift of the theta

[8] This shift can be naturally understood by embedding the 6d SCFT into M-theory. Namely, to obtain the D4-D8-08 which realizes 5d SYM description, one has to compactify the M5-M9 system on a circle with a Wilson line that breaks E_8 to $SO(16)$.

functions

$$\theta_i(m_8) \to \pm \left(\frac{y_8}{q}\right) \theta_i(m_8)\,, \tag{3.91}$$

where I have $(-)$ sign for $i = 1, 4$ and $(+)$ sign for $i = 2, 3$. The overall factor shifts by $\frac{y_8}{q}$ can be absorbed by the redefinition of the string winding fugacity $w \to wqy_8^{-1}$ [88]. To test the duality between 5d and 6d, I have to turn on the $SO(20)$ Wilson line(or its $SO(16+4N)$ generalization).

One-string

The matter contents of the 2d gauge theory description for the self-dual strings are given in Figure 3.6. I am considering $N = 1$ case, so there is an additional fundamental hypermultiplet contribution compared to the E-string theory. To compare the 6d index with the 5d index, I will study the q-expanded form of the elliptic genera finally. One-string elliptic genus is similar with E-string case [88]

$$Z^{n=1}_{\text{s.d.}} = \frac{1}{2}\left(Z_{1,[1]} + Z_{1,[2]} + Z_{1,[3]} + Z_{1,[4]}\right)\,, \tag{3.92}$$

where $Z_{1,[I]}$ are given by

$$Z_{1,[I]} = -\frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \cdot \prod_{l=1}^{10} \frac{\theta_I(m_l)}{\eta} \cdot \frac{\eta^2}{\theta_I(\epsilon_+ \pm \alpha_1)}\,. \tag{3.93}$$

The $SO(20)$ Wilson line shift will change the sign of $Z_{1,[1]}$ and $Z_{1,[4]}$. After turning on $SO(20)$ Wilson line and redefining the string winding fugacity $w \to wqy_8^{-1}$, elliptic genus of one-string is given by

$$Z^{n=1}_{\text{s.d.}} = \frac{1}{2}\left(-Z_{1,[1]} + Z_{1,[2]} + Z_{1,[3]} - Z_{1,[4]}\right)\,, \tag{3.94}$$

and the q expansion of this index is given by

$$\begin{aligned}
Z_{\text{s.d.}}^{n=1} &= q^0 \frac{t}{(1-tu)(1-t/u)} \left(\chi_{20}^{SO(20)}(y_l) - (v_1 + \frac{1}{v_1})(t + \frac{1}{t}) \right) \\
&\quad + q^1 \frac{t}{(1-tu)(1-t/u)} \frac{t^2}{(1-t^2v_1^2)(1-t^2/v_1^2)} \\
&\quad \times \left((t + \frac{1}{t}) \chi_{\overline{512}}^{SO(20)}(y_l) - (v_1 + \frac{1}{v_1}) \chi_{512}^{SO(20)}(y_l) \right) + \mathcal{O}(q^2) \\
&\equiv q^0 f_1(t,u,v,y_l) + q^1 \; Z_{\text{inst}}^1 + \mathcal{O}(q^2) \,, \qquad (3.95)
\end{aligned}$$

where f_1 and Z_{inst}^1 are defined by

$$f_1(t,u,v,y_l) = \frac{t}{(1-tu)(1-t/u)} \left(\chi_{20}^{SO(20)}(y_l) - (v_1 + \frac{1}{v_1})(t + \frac{1}{t}) \right) \,, \qquad (3.96)$$

$$\begin{aligned}
Z_1^{\text{inst}} &= \frac{t}{(1-tu)(1-t/u)} \frac{t^2}{(1-t^2v_1^2)(1-t^2/v_1^2)} \\
&\quad \times \left((t + \frac{1}{t}) \chi_{\overline{512}}^{SO(20)}(y_l) - (v_1 + \frac{1}{v_1}) \chi_{512}^{SO(20)}(y_l) \right) \,. \qquad (3.97)
\end{aligned}$$

Two-strings

Two-string elliptic genus is given by the sum of 7 discrete sectors

$$Z_{\text{s.d.}}^{n=2} = \frac{1}{2} Z_{2,[0]} + \frac{1}{4} \left(Z_{2,[1]} + Z_{2,[2]} + Z_{2,[3]} + Z_{2,[4]} + Z_{2,[5]} + Z_{2,[6]} \right) \,, \qquad (3.98)$$

where $Z_{2,[I]}$ are given by

$$\begin{aligned}
Z_{2,[0]} &= \oint \eta^2 du \frac{\theta_1(2\epsilon_+)}{i\eta} \cdot \frac{\eta^6}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_1 \pm 2u)\theta_1(\epsilon_2 \pm 2u)} \\
&\quad \times \prod_{l=1}^{10} \frac{\theta_1(m_l \pm u)}{\eta} \cdot \frac{\eta^4}{\theta_1(\epsilon_+ \pm \alpha_1 \pm u)} \,, \qquad (3.99)
\end{aligned}$$

$$\begin{aligned}
Z_{2,[I]} &= \frac{\theta_1(a_v)\theta_1(2\epsilon_+ + a_v)}{\eta^2} \cdot \frac{\eta^6}{\theta_1(\epsilon_1 + a_v)\theta_1(\epsilon_2 + a_v)\theta_1(\epsilon_1 + 2a_\pm)\theta_1(\epsilon_2 + 2a_\pm)} \\
&\quad \times \prod_{l=1}^{10} \frac{\theta_1(m_l + a_+)\theta_1(m_l + a_-)}{\eta^2} \cdot \frac{\eta^4}{\theta_1(\epsilon_+ \pm \alpha_1 + a_+)\theta_1(\epsilon_+ \pm \alpha_1 + a_-)} \,,
\end{aligned}$$

$$(3.100)$$

for $I=1,\dots,6$. Here $a_+, a_-, a_v(=a_+ + a_-)$ are given for $I=1,\dots,6$ by

$$\begin{aligned}
&[I=1]:(a_+,a_-)=(0,\frac{1}{2})\,, && [I=2]:(a_+,a_-)=(\frac{\tau}{2},\frac{1+\tau}{2})\,,\\
&[I=3]:(a_+,a_-)=(0,\frac{\tau}{2})\,, && [I=4]:(a_+,a_-)=(\frac{1}{2},\frac{1+\tau}{2})\,,\\
&[I=5]:(a_+,a_-)=(0,\frac{1+\tau}{2})\,, && [I=6]:(a_+,a_-)=(\frac{1}{2},\frac{\tau}{2})\,.
\end{aligned} \tag{3.101}$$

$Z_{2,[0]}$ has a contour integral given by JK-residue [81, 82]. The JK-residue prescription requires to sum over the residues at $u=-\frac{\epsilon_{1,2}}{2},\ -\frac{\epsilon_{1,2}}{2}+\frac{1}{2},\ -\frac{\epsilon_{1,2}}{2}+\frac{\tau}{2}, -\frac{\epsilon_{1,2}}{2}+\frac{1+\tau}{2}$ from the symmetric and $u=-\epsilon_+\pm\alpha_1$ from the fundamental hypermultiplet. The $SO(20)$ Wilson line shift changes the sign of $Z_{2,[I=1,2,5,6]}$

$$Z_{\text{s.d.}}^{n=2}=\frac{1}{2}Z_{2,[0]}+\frac{1}{4}\left(-Z_{2,[1]}-Z_{2,[2]}+Z_{2,[3]}+Z_{2,[4]}-Z_{2,[5]}-Z_{2,[6]}\right)\,, \tag{3.102}$$

again after redefining the string winding fugacity $w\to wqy_8^{-1}$. $Z_{2,[I]}$ are obtained by

$$\begin{aligned}
Z_{2,[0]}=&\frac{1}{\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)}\Bigg[\sum_{i=1}^{4}\left(\frac{1}{2}\frac{\prod_{l=1}^{10}\theta_i(m_l\pm\frac{\epsilon_1}{2})}{\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_i(\epsilon_+\pm\alpha_1\pm\frac{\epsilon_1}{2})}+(\epsilon_1\leftrightarrow\epsilon_2)\right)\\
&+\left(\frac{\prod_{l=1}^{10}\theta_1(m_l\pm(\epsilon_++\alpha_1))}{\theta_1(\epsilon_1\pm2(\epsilon_++\alpha_1))\theta_1(\epsilon_2\pm2(\epsilon_++\alpha_1))\theta_1(-2\alpha_1)\theta_1(2\epsilon_++2\alpha_1)}+(\alpha_1\to-\alpha_1)\right)\Bigg]\,,
\end{aligned} \tag{3.103}$$

$$
\begin{aligned}
Z_{2,[1]} &= \frac{\theta_2(0)\theta_2(2\epsilon_+)\prod_{l=1}^{10}\theta_1(m_l)\theta_2(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_2(\epsilon_1)\theta_2(\epsilon_2)\theta_1(\epsilon_+\pm\alpha_1)\theta_2(\epsilon_+\pm\alpha_1)}\,,\\
Z_{2,[2]} &= \frac{\theta_2(0)\theta_2(2\epsilon_+)\prod_{l=1}^{10}\theta_3(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_2(\epsilon_1)\theta_2(\epsilon_2)\theta_3(\epsilon_+\pm\alpha_1)\theta_4(\epsilon_+\pm\alpha_1)}\,,\\
Z_{2,[3]} &= \frac{\theta_4(0)\theta_4(2\epsilon_+)\prod_{l=1}^{10}\theta_1(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_4(\epsilon_1)\theta_4(\epsilon_2)\theta_1(\epsilon_+\pm\alpha_1)\theta_4(\epsilon_+\pm\alpha_1)}\,,\\
Z_{2,[4]} &= \frac{\theta_4(0)\theta_4(2\epsilon_+)\prod_{l=1}^{10}\theta_2(m_l)\theta_3(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_4(\epsilon_1)\theta_4(\epsilon_2)\theta_2(\epsilon_+\pm\alpha_1)\theta_3(\epsilon_+\pm\alpha_1)}\,,\\
Z_{2,[5]} &= \frac{\theta_3(0)\theta_3(2\epsilon_+)\prod_{l=1}^{10}\theta_1(m_l)\theta_3(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_3(\epsilon_1)\theta_3(\epsilon_2)\theta_1(\epsilon_+\pm\alpha_1)\theta_3(\epsilon_+\pm\alpha_1)}\,,\\
Z_{2,[6]} &= \frac{\theta_3(0)\theta_3(2\epsilon_+)\prod_{l=1}^{10}\theta_2(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_3(\epsilon_1)\theta_3(\epsilon_2)\theta_2(\epsilon_+\pm\alpha_1)\theta_4(\epsilon_+\pm\alpha_1)}\,.
\end{aligned}
\tag{3.104}
$$

Finally q-expanded form of the two-string elliptic genus (3.102) is

$$
\begin{aligned}
Z_{\text{s.d.}}^{n=2} = q^0\Bigg[&-\frac{t(t+\frac{1}{t})}{(1-tu)(1-t/u)} + \frac{1}{2}\left(\frac{t\left(\chi_{20}^{SO(20)}(y_l) - (v_1+\frac{1}{v_1})(t+\frac{1}{t})\right)}{(1-tu)(1-t/u)}\right)^2 \\
&+ \frac{1}{2}\left(\frac{t^2\left(\chi_{20}^{SO(20)}(y_l^2) - (v_1^2+\frac{1}{v_1^2})(t^2+\frac{1}{t^2})\right)}{(1-t^2u^2)(1-t^2/u^2)}\right)\Bigg] \\
+ q^1\Bigg[&\frac{t}{(1-tu)(1-t/u)}\frac{t^2}{(1-t^2v_1^2)(1-t^2/v_1^2)} \\
&\times\left((t+\frac{1}{t})(v_1+\frac{1}{v_1})\chi_{\overline{512}}^{SO(20)}(y_l) - (t+\frac{1}{t})^2\chi_{512}^{SO(20)}(y_l)\right) \\
&+\frac{t}{(1-tu)(1-t/u)}\left(\chi_{20}^{SO(20)}(y_l) - (v_1+\frac{1}{v_1})(t+\frac{1}{t})\right) \\
&\times\left((t+\frac{1}{t})\chi_{\overline{512}}^{SO(20)}(y_l) - (v_1+\frac{1}{v_1})\chi_{512}^{SO(20)}(y_l)\right)\Bigg] + \mathcal{O}(q^2) \\
\equiv q^0 &\left(f_2(t,u,v_1,y_l) + \frac{1}{2}\left(f_1(t,u,v_1,y_l)^2 + f_1(t^2,u^2,v_1^2,y_l^2)\right)\right) \\
+ q^1 &\left(Z_2^{\text{inst}} + f_1(t,u,v_1,y_l)Z_1^{\text{inst}}\right) + \mathcal{O}(q^2)\,,
\end{aligned}
\tag{3.105}
$$

where $f_2(t, u, v_1, y_l)$ and Z_2^{inst} are defined by

$$f_2(t, u, v_1, y_l) = -\frac{t(t + \frac{1}{t})}{(1 - tu)(1 - t/u)}, \tag{3.106}$$

$$Z_2^{\text{inst}} = \frac{t}{(1 - tu)(1 - t/u)} \frac{t^2}{(1 - t^2 v_1^2)(1 - t^2/v_1^2)} \times \left((t + \frac{1}{t})(v_1 + \frac{1}{v_1})\chi_{512}^{SO(20)}(y_l) - (t + \frac{1}{t})^2 \chi_{512}^{SO(20)}(y_l) \right). \tag{3.107}$$

Three-strings

Three-string elliptic genus is given by the sum of 8 discrete sectors which are given by

$$Z_{3,[1]} = -\oint \eta^2 du \frac{\theta_1(2\epsilon_+)\theta_1(2\epsilon_+ \pm u)\theta_1(\pm u)}{i\eta^5} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2 \theta_1(\epsilon_{1,2} \pm u)\theta_1(\epsilon_{1,2} \pm 2u)} \cdot \prod_{l=1}^{10} \frac{\theta_1(m_l)\theta_1(m_l \pm u)}{\eta^3} \cdot \frac{\eta^6}{\theta_1(\epsilon_+ \pm \alpha_1)\theta_1(\epsilon_+ \pm \alpha_1 \pm u)}, \tag{3.108}$$

$$Z_{3,[2]} = -\oint \eta^2 du \frac{\theta_1(2\epsilon_+)\theta_2(2\epsilon_+ \pm u)\theta_2(\pm u)}{i\eta^5} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2 \theta_2(\epsilon_{1,2} \pm u)\theta_1(\epsilon_{1,2} \pm 2u)} \cdot \prod_{l=1}^{10} \frac{\theta_2(m_l)\theta_1(m_l \pm u)}{\eta^3} \cdot \frac{\eta^6}{\theta_2(\epsilon_+ \pm \alpha_1)\theta_1(\epsilon_+ \pm \alpha_1 \pm u)}, \tag{3.109}$$

$$Z_{3,[3]} = -\oint \eta^2 du \frac{\theta_1(2\epsilon_+)\theta_3(2\epsilon_+ \pm u)\theta_3(\pm u)}{i\eta^5} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2 \theta_3(\epsilon_{1,2} \pm u)\theta_1(\epsilon_{1,2} \pm 2u)} \cdot \prod_{l=1}^{10} \frac{\theta_3(m_l)\theta_1(m_l \pm u)}{\eta^3} \cdot \frac{\eta^6}{\theta_3(\epsilon_+ \pm \alpha_1)\theta_1(\epsilon_+ \pm \alpha_1 \pm u)}, \tag{3.110}$$

$$Z_{3,[4]} = -\oint \eta^2 du \frac{\theta_1(2\epsilon_+)\theta_4(2\epsilon_+ \pm u)\theta_4(\pm u)}{i\eta^5} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2 \theta_4(\epsilon_{1,2} \pm u)\theta_1(\epsilon_{1,2} \pm 2u)} \cdot \prod_{l=1}^{10} \frac{\theta_4(m_l)\theta_1(m_l \pm u)}{\eta^3} \cdot \frac{\eta^6}{\theta_4(\epsilon_+ \pm \alpha_1)\theta_1(\epsilon_+ \pm \alpha_1 \pm u)}, \tag{3.111}$$

(3.112)

$$
\begin{aligned}
Z_{3,[I']} = &-\frac{\prod_{i>j}^{3}\theta_1(a_i+a_j)\theta_1(2\epsilon_+ + a_i+a_j)}{\eta^6}\cdot\frac{\eta^{12}}{\prod_{i\geq j}^{3}\theta_1(\epsilon_{1,2}+a_i+a_j)} \\
&\cdot\prod_{l=1}^{10}\frac{\prod_{i=1}^{3}\theta_1(m_l+a_i)}{\eta^3}\cdot\frac{\eta^6}{\prod_{i=1}^{3}\theta_1(\epsilon_+\pm\alpha_1+a_i)}\,,
\end{aligned}
\tag{3.113}
$$

where a_1, a_2, a_3 are given for $I' = 1', 2', 3', 4'$ by

$$
\begin{aligned}
&[I'=1'] : (a_1,a_2,a_3) = (\frac{1}{2},\frac{1+\tau}{2},\frac{\tau}{2})\ , \quad [I'=2'] : (a_1,a_2,a_3) = (\frac{\tau}{2},\frac{1+\tau}{2},0)\ , \\
&[I'=3'] : (a_1,a_2,a_3) = (0,\frac{\tau}{2},\frac{1}{2})\ , \qquad\ \ [I'=4'] : (a_1,a_2,a_3) = (\frac{1}{2},\frac{1+\tau}{2},0)\ .
\end{aligned}
\tag{3.114}
$$

Each $Z_{3,[I]}$ has a contour integral. The non-zero JK-residues come from the poles at $u = -\frac{\epsilon_{1,2}}{2},\ -\frac{\epsilon_{1,2}}{2}+\frac{1}{2},\ -\frac{\epsilon_{1,2}}{2}+\frac{\tau}{2}, -\frac{\epsilon_{1,2}}{2}+\frac{1+\tau}{2}, -\epsilon\pm\alpha$ and $u = -\epsilon_{1,2}+\cdots$, where $\cdots$ part is decided by $\theta_i(\epsilon_{1,2}+u) = 0$. After turning on the $SO(20)$ Wilson line, the three-string elliptic genus becomes

$$
\begin{aligned}
Z^{n=3}_{\text{s.d.}} &= \frac{1}{4}\left(-Z_{3,[1]}+Z_{3,[2]}+Z_{3,[3]}-Z_{3,[4]}\right)+\frac{1}{8}\left(-Z_{3,[1']}+Z_{3,[2']}+Z_{3,[3']}-Z_{3,[4']}\right) \\
&= q^0\Bigg(\frac{1}{3}f_1(t^3,u^3,v_1^3,y_l^3)+\frac{1}{6}f_1(t,u,v_1,y_l)^3 \\
&\qquad +\frac{1}{2}f_1(t,u,v_1,y_l)f_1(t^2,u^2,v_1^2,y_u^2)+f_1(t,u,v_1,y_l)f_2(t,u,v_1,y_l)\Bigg) \\
&\quad + q\Bigg((Z_3^{\text{inst}}+f_1(t,u,v_1,y_l)Z_2^{\text{inst}} \\
&\qquad +\left(f_2(t,u,v_1,y_l)+\frac{1}{2}\left(f_1(t,u,v_1,y_l)^2+f_1(t^2,u^2,v_1^2,y_l^2)\right)\right)Z_1^{\text{inst}}\Bigg)+\mathcal{O}(q^2)\,,
\end{aligned}
\tag{3.115}
$$

where Z_3^{inst} are defined by

$$Z_3^{\text{inst}} = \frac{t}{(1-tu)(1-t/u)} \frac{t^2}{(1-t^2v_1^2)(1-t^2/v_1^2)} \\ \times \left((t+\frac{1}{t})(t^2+1+\frac{1}{t^2})\chi_{\overline{512}}^{SO(20)}(y_l) - (v_1+\frac{1}{v_1})(t^2+1+\frac{1}{t^2})\chi_{512}^{SO(20)}(y_l) \right) . \tag{3.116}$$

Perturbative index

The perturbative index of the theory on a circle is given by

$$Z_{\text{pert}}^{\text{6d},Sp(1)} = \text{PE}\Bigg[\left(\frac{t}{(1-tu)(1-t/u)} \right) \left(-(t+\frac{1}{t}) \left(\chi_{\text{adj},+}^{Sp(1)} + \chi_{\text{adj}}^{Sp(1)} \frac{q^2}{1-q^2} \right) \right. \\ \left. + \left(\chi_{\text{fund},+}^{Sp(1)} \chi_{\text{fund}}^{SO(20)} + \chi_{\text{fund}}^{Sp(1)} \chi_{\text{fund}}^{SO(20)} \frac{q^2}{1-q^2} \right) \right) \Bigg] , \tag{3.117}$$

where PE is defined in (3.58). First term of the index comes from the 6d W-bosons and second term comes from the 6d fundamental quarks. The background $SO(20)$ Wilson line has no effect on the fields in the $SO(20)$ fundamental representation, and only affects spinor representation. So the perturbative index is unaffected by this Wilson line. In the exponent, I have only kept the contributions from BPS states with positive central charges in the regime $q \ll v_1 \ll y_l^{\pm 1}$.

Full index

After combining all ingredients, BPS index for the 6d $Sp(1)$ SCFT with $N_f = 10$ matters is written as

$$\begin{aligned} Z^{\text{6d},Sp(1)} &= Z_{\text{pert}}^{\text{6d},Sp(1)} Z_{\text{s.d}}^{\text{6d},Sp(1)} \\ &= Z_{\text{pert}}^{\text{6d},Sp(1)} \left(1 + w^1 Z_{\text{s.d.}}^{n=1} + w^2 Z_{\text{s.d.}}^{n=2} + w^3 Z_{\text{s.d.}}^{n=3} + \mathcal{O}(w^4)\right) , \end{aligned} \tag{3.118}$$

where all ingredients are given in (3.95), (3.105), (3.115), (3.117).

3.2.3 Duality test

I want to compare the BPS indices of 5d $Sp(2)$ gauge theory and 6d $Sp(1)$ SCFT to test their duality. If I expand the 5d index $Z^{5\text{d},Sp(2)}$ given in (3.59) in terms of one of the $Sp(2)$ fugacity v_2 and set $v_2 = w$, then it is exactly same as the 6d BPS index $Z^{6\text{d},Sp(1)}$ which is given (3.118) in the sense of double expansion of the instanton fugacity q and the string winding fugacity w. The one of the 5d Coulomb VEV fugacity v_2 is identified the 6d string winding number fugacity w, and the instanton fugacity q becomes the string momentum fugacity q. Keeping in mind the 5d-6d fugacity relations and the E_8 or $SO(20)$ Wilson line effect, I will study the 6d $Sp(N)$ gauge theories and their 5d $Sp(N+1)$ gauge theory descriptions.

3.2.4 $Sp(N)$ generalization

In the above, I observed that the 6d string winding fugacity w corresponds to one of the fugacities for the 5d $Sp(2)$ gauge symmetry in the instanton partition function. I can generalize this observation. $Sp(N+1)$ group can be decomposed into $Sp(1) \times Sp(N) \subset Sp(N+1)$. I expect that the former $Sp(1) \sim SU(2)$ is responsible for the string winding fugacity, and the latter $Sp(N)$ gives the 6d gauge symmetry. I will confirm this assertion by comparing the 5d and 6d indices.

5d $Sp(N+1)$ gauge theory index

The 5d $Sp(2)$ index given in (3.59) can be easily generalized for $Sp(N+1)$ gauge group and $N_f = 2N+8$ fundamental hypermultiplets. This is given by

$$
\begin{aligned}
&Z^{5\text{d},Sp(N+1)} \\
&= \mathrm{PE}\left[\frac{t}{(1-tu)(1-t/u)}\left(-(t+\frac{1}{t})\chi^{Sp(N+1)}_{\text{adj},+} + \chi^{Sp(N+1)}_{\text{fund},+}\chi^{SO(4N+16)}_{\text{fund}}\right)\right] \\
\times &\left(1 + q\left(\frac{\prod_{l=1}^{2N+8} 2\sinh\frac{m_l}{2}}{2\sinh\frac{\epsilon_1}{2}2\sinh\frac{\epsilon_2}{2}\prod_{i=1}^{N+1}2\sinh\frac{\epsilon_+ \pm \alpha_i}{2}} + \frac{\prod_{l=1}^{2N+8} 2\cosh\frac{m_l}{2}}{2\cosh\frac{\epsilon_1}{2}2\cosh\frac{\epsilon_2}{2}\prod_{i=1}^{N+1}2\cosh\frac{\epsilon_+ \pm \alpha_i}{2}}\right)\right. \\
&\qquad \left. + \mathcal{O}(q^2)\right). \qquad (3.119)
\end{aligned}
$$

First line is the perturbative index and second line is the one-instanton partition function. To compare this result with the 6d index, I specially treat one of the Coulomb VEV fugacity $v_{N+1} = e^{-\alpha_{N+1}} \equiv w$. Then $Sp(N+1)$ characters can be rewritten in terms of $Sp(N)$ characters and w

$$
\chi^{Sp(N+1)}_{\text{fund}}(v_i) \equiv \sum_{i=1}^{N+1}\left(v_i + \frac{1}{v_i}\right) = \chi^{Sp(N)}_{\text{fund}}(v_i) + \left(w + \frac{1}{w}\right), \qquad (3.120)
$$

$$
\begin{aligned}
\chi^{Sp(N+1)}_{\text{adj}}(v_i) &\equiv \frac{\left(\chi^{Sp(N+1)}_{\text{fund}}(v_i)\right)^2 + \chi^{Sp(N+1)}_{\text{fund}}(v_i^2)}{2} \\
&= \frac{\left(\chi^{Sp(N)}_{\text{fund}}(v_i) + \left(w+\frac{1}{w}\right)\right)^2 + \chi^{Sp(N)}_{\text{fund}}(v_i^2) + \left(w^2 + \frac{1}{w^2}\right)}{2} \\
&= \chi^{Sp(N)}_{\text{adj}}(v_i) + \left(w + \frac{1}{w}\right)\chi^{Sp(N)}_{\text{fund}}(v_i) + w^2 + 1 + \frac{1}{w^2}. \qquad (3.121)
\end{aligned}
$$

Then the perturbative index becomes

$$
\begin{aligned}
&Z^{5\text{d},Sp(N+1)}_{\text{pert}} \\
&= \mathrm{PE}\left[\frac{t}{(1-tu)(1-t/u)}\left(-(t+\frac{1}{t})\chi^{Sp(N)}_{\text{adj},+} + \chi^{Sp(N)}_{\text{fund},+}\chi^{SO(4N+16)}_{\text{fund}}\right)\right] \\
&\times \mathrm{PE}\left[\frac{t}{(1-tu)(1-t/u)}\left(-(t+\frac{1}{t})w^2 + w\left(-(t+\frac{1}{t})\chi^{Sp(N)}_{\text{fund}} + \chi^{SO(4N+16)}_{\text{fund}}\right)\right)\right], \\
&\qquad (3.122)
\end{aligned}
$$

where I only keep positive weights(roots) in the plethystic exponential.

One can expand the instanton partition function in terms of w

$$
\begin{aligned}
&Z^{k=1}_{5\mathrm{d},Sp(N+1)}\\
&=\frac{1}{2}\left(\frac{\prod_{l=1}^{2N+8}2\sinh\frac{m_l}{2}}{2\sinh\frac{\epsilon_1}{2}2\sinh\frac{\epsilon_2}{2}\prod_{i=1}^{N}2\sinh\frac{\epsilon_+\pm\alpha_i}{2}}\frac{t}{(1-tw)(1-t/w)}\right.\\
&\qquad\left.+\frac{\prod_{l=1}^{2N+8}2\cosh\frac{m_l}{2}}{2\cosh\frac{\epsilon_1}{2}2\cosh\frac{\epsilon_2}{2}\prod_{i=1}^{N}2\cosh\frac{\epsilon_+\pm\alpha_i}{2}}\frac{t}{(1+tw)(1+t/w)}\right)\\
&=\frac{1}{2}w\left(-\frac{\prod_{l=1}^{2N+8}2\sinh\frac{m_l}{2}}{2\sinh\frac{\epsilon_1}{2}2\sinh\frac{\epsilon_2}{2}\prod_{i=1}^{N}2\sinh\frac{\epsilon_+\pm\alpha_i}{2}}\right.\\
&\qquad\left.+\frac{\prod_{l=1}^{2N+8}2\cosh\frac{m_l}{2}}{2\cosh\frac{\epsilon_1}{2}2\cosh\frac{\epsilon_2}{2}\prod_{i=1}^{N}2\cosh\frac{\epsilon_+\pm\alpha_i}{2}}\right)\\
&\quad-\frac{1}{2}w^2\left(t+\frac{1}{t}\right)\left(\frac{\prod_{l=1}^{2N+8}2\sinh\frac{m_l}{2}}{2\sinh\frac{\epsilon_1}{2}2\sinh\frac{\epsilon_2}{2}\prod_{i=1}^{N}2\sinh\frac{\epsilon_+\pm\alpha_i}{2}}\right.\\
&\qquad\left.+\frac{\prod_{l=1}^{2N+8}2\cosh\frac{m_l}{2}}{2\cosh\frac{\epsilon_1}{2}2\cosh\frac{\epsilon_2}{2}\prod_{i=1}^{N}2\cosh\frac{\epsilon_+\pm\alpha_i}{2}}\right)\\
&\quad+\frac{1}{2}w^3\left(t^2+1+\frac{1}{t^2}\right)\left(-\frac{\prod_{l=1}^{2N+8}2\sinh\frac{m_l}{2}}{2\sinh\frac{\epsilon_1}{2}2\sinh\frac{\epsilon_2}{2}\prod_{i=1}^{N}2\sinh\frac{\epsilon_+\pm\alpha_i}{2}}\right.\\
&\qquad\left.+\frac{\prod_{l=1}^{2N+8}2\cosh\frac{m_l}{2}}{2\cosh\frac{\epsilon_1}{2}2\cosh\frac{\epsilon_2}{2}\prod_{i=1}^{N}2\cosh\frac{\epsilon_+\pm\alpha_i}{2}}\right)\\
&\quad+\cdots\,.
\end{aligned}
\tag{3.123}
$$

Now I will compare this with the 6d index. Note that the first line of (3.122) is already same as the 6d perturbative index, so w^0q^0 orders clearly agree with each other.

6d $Sp(N)$ SCFT index

Firstly, I compare the 5d-6d results at w^1q^0 and w^1q^1 orders. One-string elliptic genus has following form

$$Z_{n=1}^{6\text{d},Sp(N)} = \frac{1}{2}\left(-Z_{1,[1]} + Z_{1,[2]} + Z_{1,[3]} - Z_{1,[4]}\right) \,, \tag{3.124}$$

where $Z_{1,[I]}$ are given by

$$Z_{1,[I]} = -\frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \prod_{i=1}^{N} \frac{\eta^2}{\theta_I(\epsilon_+ \pm \alpha_i)} \prod_{l=1}^{2N+8} \frac{\theta_I(m_l)}{\eta} \,. \tag{3.125}$$

After making q expansion of $Z_{1,[I]}$, and after replacing all chemical potential by $z \to \frac{iz}{2\pi}$ (where z denotes $\epsilon_{1,2}, \alpha_i, m_l$), one obtains

$$Z_{1,[1]} = \frac{\prod_{l=1}^{2N+8} 2\sinh\frac{m_l}{2}}{2\sinh\frac{\epsilon_1}{2} 2\sinh\frac{\epsilon_2}{2} \prod_{i=1}^{N} 2\sinh\frac{\epsilon_+ \pm \alpha_i}{2}} q^1 + \mathcal{O}(q^2)\,, \tag{3.126}$$

$$Z_{1,[2]} = \frac{\prod_{l=1}^{2N+8} 2\cosh\frac{m_l}{2}}{2\cosh\frac{\epsilon_1}{2} 2\cosh\frac{\epsilon_2}{2} \prod_{i=1}^{N} 2\cosh\frac{\epsilon_+ \pm \alpha_i}{2}} q^1 + \mathcal{O}(q^2)\,, \tag{3.127}$$

$$Z_{1,[3]} = \left(\frac{\sum_{l=1}^{2N+8} 2\cosh\frac{m_l}{2} - \sum_{i=1}^{N} 2\cosh\frac{\epsilon_+ \pm \alpha_i}{2}}{2 \cdot 2\sinh\frac{\epsilon_1}{2} 2\sinh\frac{\epsilon_2}{2}}\right) q^0 + F(m_l, \alpha_i, \epsilon_i) q^1 + \mathcal{O}(q^2)\,, \tag{3.128}$$

$$Z_{1,[4]} = -\left(\frac{\sum_{l=l}^{2N+8} 2\cosh\frac{m_l}{2} - \sum_{i=1}^{N} 2\cosh\frac{\epsilon_+ \pm \alpha_i}{2}}{2 \cdot 2\sinh\frac{\epsilon_1}{2} 2\sinh\frac{\epsilon_2}{2}}\right) q^0 + F(m_l, \alpha_i, \epsilon_i) q^1 + \mathcal{O}(q^2)\,. \tag{3.129}$$

I didn't write explicit form of $F(m_l, \alpha_i, \epsilon_i)$ which is the coefficient of q^1 in $Z_{1,[3]}$ and $Z_{1,[4]}$, because they are canceled after summation. Then $w^1 q^0$ term in (3.124) agrees with (3.122). Also I have checked that $w^1 q^1$ term agrees with the corresponding order of Z^{5d}.

Next, I compare the 5d-6d results at $w^2 q^0$ and $w^2 q^1$ orders. Two-string elliptic genus is given by

$$
\begin{aligned}
Z_{2,[0]} =& \oint \eta^2 du \frac{\theta_1(2\epsilon_+)}{i\eta} \cdot \frac{\eta^6}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_1 \pm 2u)\theta_1(\epsilon_2 \pm 2u)} \\
&\cdot \prod_{l=1}^{2N+8} \frac{\theta_1(m_l \pm u)}{\eta} \cdot \prod_{i=1}^{N} \frac{\eta^4}{\theta_1(\epsilon_+ \pm \alpha_i \pm u)}\,, \\
Z_{2,[I]} =& \frac{\theta_1(a_v)\theta_1(2\epsilon_+ + a_v)}{\eta^2} \cdot \frac{\eta^6}{\theta_1(\epsilon_1 + a_v)\theta_1(\epsilon_2 + a_v)\theta_1(\epsilon_1 + 2a_\pm)\theta_1(\epsilon_2 + 2a_\pm)} \\
&\cdot \prod_{l=1}^{2N+8} \frac{\theta_1(m_l + a_+)\theta_1(m_l + a_-)}{\eta^2} \cdot \prod_{i=1}^{N} \frac{\eta^4}{\theta_1(\epsilon_+ \pm \alpha_i + a_+)\theta_1(\epsilon_+ \pm \alpha_i + a_-)}\,,
\end{aligned}
\tag{3.130}
$$

where discrete sector I is same as (3.114). There are additional poles from symmetric hypermultiplets, which are given by $u_* = -\epsilon_+ \pm \alpha_i$ for all i. Now I can obtain general form of two-strings elliptic genus

$$
\begin{aligned}
Z_{2,[0]} =& \frac{1}{\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \left[\sum_{i=1}^{4} \left(\frac{1}{2} \frac{\prod_{l=1}^{2N+8} \theta_i(m_l \pm \frac{\epsilon_1}{2})}{\theta_1(2\epsilon_1)\theta_1(\epsilon_2 - \epsilon_1) \prod_{m=1}^{N} \theta_i(\epsilon_+ \pm \alpha_m \pm \frac{\epsilon_1}{2})} + (\epsilon_1 \leftrightarrow \epsilon_2) \right) \right. \\
&+ \sum_{n=1}^{N} \left(\frac{\prod_{l=1}^{2N=8} \theta_1(m_l \pm (\epsilon_+ + \alpha_n))}{\theta_1(\epsilon_{1,2} \pm 2(\epsilon_+ + \alpha_n))\theta_1(-2\alpha_n)\theta_1(2\epsilon_+ + 2\alpha_n) \prod_{m\neq n}^{N} \theta_1(-\alpha_n \pm \alpha_m)\theta_1(2\epsilon_+ + \alpha_n \pm \alpha_m)} \right. \\
&\left.\left. + (\alpha_n \to -\alpha_n) \right) \right],
\end{aligned}
\tag{3.131}
$$

$$
\begin{aligned}
Z_{2,[1]} &= \frac{\theta_2(0)\theta_2(2\epsilon_+)\prod_{l=1}^{2N+8}\theta_1(m_l)\theta_2(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_2(\epsilon_1)\theta_2(\epsilon_2)\prod_{m=1}^{N}\theta_1(\epsilon_+\pm\alpha_m)\theta_2(\epsilon_+\pm\alpha_m)}\,,\\
Z_{2,[2]} &= \frac{\theta_2(0)\theta_2(2\epsilon_+)\prod_{l=1}^{2N+8}\theta_3(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_2(\epsilon_1)\theta_2(\epsilon_2)\prod_{m=1}^{N}\theta_3(\epsilon_+\pm\alpha_m)\theta_4(\epsilon_+\pm\alpha_m)}\,,\\
Z_{2,[3]} &= \frac{\theta_4(0)\theta_4(2\epsilon_+)\prod_{l=1}^{2N+8}\theta_1(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_4(\epsilon_1)\theta_4(\epsilon_2)\prod_{m=1}^{N}\theta_1(\epsilon_+\pm\alpha_m)\theta_4(\epsilon_+\pm\alpha_m)}\,,\\
Z_{2,[4]} &= \frac{\theta_4(0)\theta_4(2\epsilon_+)\prod_{l=1}^{2N+8}\theta_2(m_l)\theta_3(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_4(\epsilon_1)\theta_4(\epsilon_2)\prod_{m=1}^{N}\theta_2(\epsilon_+\pm\alpha_m)\theta_3(\epsilon_+\pm\alpha_m)}\,,\\
Z_{2,[5]} &= \frac{\theta_3(0)\theta_3(2\epsilon_+)\prod_{l=1}^{2N+8}\theta_1(m_l)\theta_3(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_3(\epsilon_1)\theta_3(\epsilon_2)\prod_{m=1}^{N}\theta_1(\epsilon_+\pm\alpha_m)\theta_3(\epsilon_+\pm\alpha_m)}\,,\\
Z_{2,[6]} &= \frac{\theta_3(0)\theta_3(2\epsilon_+)\prod_{l=1}^{2N+8}\theta_2(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_3(\epsilon_1)\theta_3(\epsilon_2)\prod_{m=1}^{N}\theta_2(\epsilon_+\pm\alpha_m)\theta_4(\epsilon_+\pm\alpha_m)}\,.
\end{aligned}
\tag{3.132}
$$

After plugging these into the (3.102), one can obtain two-string elliptic genus. I compared the q-expanded form of this elliptic genus with 5d index by increasing N up to $N = 8$, and I saw perfect agreements of the two results. I also checked the agreement of three-string elliptic genus up to $N = 3$.

3.3 Conclusion

I tested the duality between 5d $Sp(N + 1)$ and $SU(N + 2)$ gauge theories and their 6d UV SCFT via their supersymmetric indices. I first compared the supersymmetric indices of the 5d and 6d theories for $N = 1$ case. The usual ADHM construction contains unwanted string theory degrees in 5d gauge theories case. I solve this problem theory by theory using the simple tricks.

For 5d $SU(3)$ theory, I can detour this problem by introducing anti-symmetric hypermultiplets. Anti-symmetric representation is same as (anti-)fundamental representation in $SU(3)$ group. So the 5d $SU(3)$ gauge theory with 10 fundamental hypermultiplets can be regarded as the gauge theory with 8 fundamental and 2 anti-symmetric hypermultiplets. This trivial change of viewpoint affects the details of the ADHM construction. Such alternative ADHM descriptions are sometimes shown to provide more useful description of instantons [52]. For 5d $Sp(2)$ theory, one-instanton sector which realized by localized half-D1-brane doesn't have extra string theory degrees. Using this fact, I compute one-instanton partition function exactly. I observe perfect agreement of the two indices in 5d gauge theories under the highly non-trivial fugacity map. This fugacity map can be read by comparing their Type IIB brane diagrams [57].

I also compute elliptic genera of the self-dual strings in 6d $Sp(1)$ SCFT with 10 fundamental flavors using the Type IIA brane description. I also see the perfect agreement of the 5d $Sp(N+1)$ and 6d $Sp(N)$ indices in double expansion of the string winding fugacity w and the instanton fugacity q. As usual, the 5d instanton charge is mapped to the 6d KK momentum mode. The 5d $Sp(N+1)$ gauge group is decomposed into the $Sp(1)\times Sp(N)$, and the former $Sp(1)$ charge is mapped the 6d self-dual string winding number. The fugacities for the latter 5d $Sp(N)$ gauge symmetry and $SO(4N+16)$ flavor symmetry are mapped to the 6d $Sp(N)$ gauge symmetry and $SO(4N+16)$ flavor symmetry. I have also observed that the background $SO(4N+16)$ Wilson line plays crucial roles in these 5d-6d dualities, similar to the E_8 Wilson line in the E-string theory. These results provide the detailed rules of the dualities proposed by [55].

The natural question is what happens if one naively compute higher-order instanton partition functions using (3.45). Naive computation shows disagreements with the result predicted by the elliptic genus of self-dual strings. Difference between two results must comes from the extra degrees in the string engineered ADHM construction. I hope this result gives the better understanding of the extra degrees in the brane system.

Another question is the application of the exchange of $SU(3)$ fundamental to anti-symmetric hypermultiplets. Naive computation shows that such replacement works up to two matters, which has brane description. I wonder why such trick works sometimes and doesn't work.

Appendix A

8-point amplitude factorization channels

A.1 Explicit computation for 8-point amplitude factorization channels

In this section, I work in the u-factorization gauge introduced in section 2.3.1 instead of u-cyclic gauge:

$$C = \begin{pmatrix} 1 & \hat{u}_{12} & \hat{u}_{13} & \hat{u}_{14} & 1 & -\hat{u}_{12} & -\hat{u}_{13} & -\hat{u}_{14} \\ -\hat{u}_{12} & 1 & \hat{u}_{23} & \hat{u}_{24} & \hat{u}_{12} & 1 & -\hat{u}_{23} & -\hat{u}_{24} \\ -\hat{u}_{13} & -\hat{u}_{23} & 1 & \hat{u}_{34} & \hat{u}_{13} & \hat{u}_{23} & 1 & -\hat{u}_{34} \\ -\hat{u}_{14} & -\hat{u}_{24} & -\hat{u}_{34} & 1 & \hat{u}_{14} & \hat{u}_{24} & \hat{u}_{34} & 1 \end{pmatrix}, \quad \hat{u}_{mn} = \hat{u}_{mn}(z)\,. \tag{A.1}$$

The minors are also quadratic polynomials in z in the u-factorization gauge. The minors are given in below and I use the shorthand notations for later convenience

$$
\begin{aligned}
M_1 &= z^2 + (\hat{u}_{12}^2 + \hat{u}_{13}^2 + \hat{u}_{14}^2 + \hat{u}_{23}^2 + \hat{u}_{24}^2 + \hat{u}_{34}^2) + 1 \\
&= (1+V)z^2 + 2Uz + (1+W)\,, \\
M_2 &= z^2 - (\hat{u}_{23}^2 + \hat{u}_{34}^2 + \hat{u}_{24}^2) + (\hat{u}_{12}^2 + \hat{u}_{13}^2 + \hat{u}_{14}^2) - 1 \\
&= (1 + r_2 - s_2)z^2 - 2(p_2 - q_2)z - (1 + m_2 - n_2)\,, \\
M_3 &= z^2 + (\hat{u}_{12}^2 + \hat{u}_{34}^2) - (\hat{u}_{13}^2 + \hat{u}_{24}^2 + \hat{u}_{14}^2 + \hat{u}_{23}^2) + 1 \\
&= (1 + r_3 - s_3)z^2 + 2(p_3 - q_3)z + (1 + m_3 - n_3)\,, \\
M_4 &= z^2 - (\hat{u}_{12}^2 + \hat{u}_{13}^2 + \hat{u}_{23}^2) + (\hat{u}_{14}^2 + \hat{u}_{24}^2 + \hat{u}_{34}^2) - 1 \\
&= (1 + r_4 - s_4)z^2 - 2(p_4 - q_4)z - (1 + m_4 - n_4)\,.
\end{aligned}
\tag{A.2}
$$

I introduced

$$
V = \frac{1}{2}\sum_{i,j}(\bar{u}_*^{ij})^2\,, \quad U = \frac{1}{4}\sum_{i,j,k,l}\epsilon_{ijkl}u_{ij}^*\bar{u}_*^{kl}\,, \quad W = \frac{1}{2}\sum_{i,j}(u_{ij}^*)^2\,,
\tag{A.3}
$$

and

$$
\begin{array}{rclrcl}
r_2 &=& (\bar{u}_*^{23})^2 + (\bar{u}_*^{34})^2 + (\bar{u}_*^{24})^2\,, & s_2 &=& (\bar{u}_*^{12})^2 + (\bar{u}_*^{13})^2 + (\bar{u}_*^{14})^2\,, \\
p_2 &=& u_{23}^*\bar{u}_*^{14} - u_{24}^*\bar{u}_*^{13} + u_{34}^*\bar{u}_*^{12}\,, & q_2 &=& u_{12}^*\bar{u}_*^{34} - u_{13}^*\bar{u}_*^{24} + u_{14}^*\bar{u}_*^{23}\,, \\
m_2 &=& (u_{23}^*)^2 + (u_{34}^*)^2 + (u_{24}^*)^2\,, & n_2 &=& (u_{12}^*)^2 + (u_{13}^*)^2 + (u_{14}^*)^2\,, \\
r_3 &=& (\bar{u}_*^{12})^2 + (\bar{u}_*^{34})^2\,, & s_3 &=& (\bar{u}_*^{13})^2 + (\bar{u}_*^{24})^2 + (\bar{u}_*^{14})^2 + (\bar{u}_*^{23})^2, \\
p_3 &=& u_{12}^*\bar{u}_*^{34} + u_{34}^*\bar{u}_*^{12}\,, & q_3 &=& u_{14}^*\bar{u}_*^{23} + u_{23}^*\bar{u}_*^{14} - u_{13}^*\bar{u}_*^{24} - u_{24}^*\bar{u}_*^{13}\,, \\
m_3 &=& (u_{12}^*)^2 + (u_{34}^*)^2\,, & n_3 &=& (u_{13}^*)^2 + (u_{14}^*)^2 + (u_{23}^*)^2 + (u_{24}^*)^2\,, \\
r_4 &=& (\bar{u}_*^{12})^2 + (\bar{u}_*^{13})^2 + (\bar{u}_*^{23})^2\,, & s_4 &=& (\bar{u}_*^{14})^2 + (\bar{u}_*^{24})^2 + (\bar{u}_*^{34})^2\,, \\
p_4 &=& u_{12}^*\bar{u}_*^{34} - u_{13}^*\bar{u}_*^{24} + u_{23}^*\bar{u}_*^{14}\,, & q_4 &=& u_{14}^*\bar{u}_*^{23} - u_{24}^*\bar{u}_*^{13} + u_{34}^*\bar{u}_*^{12}\,, \\
m_4 &=& (u_{12}^*)^2 + (u_{13}^*)^2 + (u_{23}^*)^2\,, & n_4 &=& (u_{14}^*)^2 + (u_{24}^*)^2 + (u_{34}^*)^2\,.
\end{array}
\tag{A.4}
$$

Note that $r_h + s_h = V$, $p_h + q_h = U$, $m_h + n_h = W$ for each $h = 2, 3, 4$. The quantities $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$ can be expressed in terms of $(m_h, n_h, p_h, q_h, r_h, s_h)$ in a simple way:

$$
\begin{aligned}
&\alpha_{12} = -4\left[(m_2+1)q_2 - n_2 p_2\right] , \quad \gamma_{12} = -4\left[p_2(r_2+1) - q_2 s_2\right] , \\
&\beta_{12} = 2\left[(m_2+1)(r_2+1) - n_2 s_2\right] , \\
&\alpha_{13} = 4\left[(m_3+1)q_3 - n_3 p_3\right] , \quad \gamma_{13} = 4\left[p_3 s_3 - q_3(r_3+1)\right] , \\
&\beta_{13} = -2\left[(m_3+1)s_3 - n_3(r_3+1)\right] , \\
&\alpha_{14} = -4\left[(m_4+1)q_4 - n_4 p_4\right] , \quad \gamma_{14} = -4\left[p_4(r_4+1) - q_4 s_4\right] , \\
&\beta_{14} = 2\left[(m_4+1)(r_4+1) - n_4 s_4\right] .
\end{aligned} \tag{A.5}
$$

I want to verify the claim made in (2.123). As explained in the main text, the two physical poles for p^2_{123} and p^2_{567} are captured by Δ_{14}. With a slight rewriting to facilitate comparison with the u-gauge, I have

$$
\begin{aligned}
\left(\frac{4}{R}\right)^2 p^2_{123} &= \left(\frac{4}{R}\right)^2 \left(\langle 12\rangle^2 + \langle 13\rangle^2 + \langle 23\rangle^2\right) \\
&= (1+r_4)(1+m_4) - 2(\bar{u}_*^{14}u^*_{14} + \bar{u}_*^{24}u^*_{24} + \bar{u}_*^{34}u^*_{34})^2 + s_4 n_4 \\
&\quad + \left((\bar{u}_*^{12} + u^*_{12}) + 2(\bar{u}_*^{23}u^*_{13} - \bar{u}_*^{13}u^*_{23})\right)(\bar{u}_*^{24}u^*_{14} - \bar{u}_*^{14}u^*_{24}) \\
&\quad + \left((\bar{u}_*^{13} + u^*_{13}) + 2(\bar{u}_*^{32}u^*_{12} - \bar{u}_*^{12}u^*_{32})\right)(\bar{u}_*^{34}u^*_{14} - \bar{u}_*^{14}u^*_{34}) \\
&\quad + \left((\bar{u}_*^{23} + u^*_{23}) + 2(\bar{u}_*^{31}u^*_{21} - \bar{u}_*^{21}u^*_{31})\right)(\bar{u}_*^{34}u^*_{24} - \bar{u}_*^{24}u^*_{34}) \\
&\equiv (1+r_4)(1+m_4) - 2(t_4)^2 + s_4 n_4 \\
&\quad + (x_{12} + 2y_{12})z_{12} + (x_{23} + 2y_{23})z_{23} + (x_{31} + 2y_{31})z_{31} ,
\end{aligned} \tag{A.6}
$$

and, similarly,

$$\left(\frac{4}{R}\right)^2 p_{567}^2 = \left(\frac{4}{R}\right)^2 (\langle 56\rangle^2 + \langle 57\rangle^2 + \langle 67\rangle^2)$$
$$= (1 + r_4)(1 + m_4) - 2(t_4)^2 + s_4 n_4$$
$$+ (-x_{12} + 2y_{12})z_{12} + (-x_{23} + 2y_{23})z_{23} + (-x_{31} + 2y_{31})z_{31} \,. \tag{A.7}$$

Combining the two, I have

$$\frac{2^8}{R^4} p_{123}^2 p_{567}^2 = \left[(1 + r_4)(1 + m_4) - 2(t_4)^2 + s_4 n_4 + 2(y_{12}z_{12} + y_{23}z_{23} + y_{31}z_{31})\right]^2$$
$$- (x_{12}z_{12} + x_{23}z_{23} + x_{31}z_{31})^2 \,. \tag{A.8}$$

On the other hand, I deduce from (A.5) that

$$\frac{1}{4}\Delta_{14} = [(m_4 + 1)(r_4 + 1) - n_4 s_4]^2 - 4\,[(m_4 + 1)q_4 - n_4 p_4]\,[p_4(r_4 + 1) - q_4 s_4] \,. \tag{A.9}$$

A lengthy but straightforward computation shows that (A.8) and (A.9) are equal.

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초록

ABJM 산란 진폭 &
고차원 초등각장이론의 이중성 확인

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이 논문은 두가지 부분으로 구성되어 있다. 첫번째 부분에서는 ABJM 산란 진폭에 관하여 논의한다. 그라스마니안 식을 이용한 ABJM 산란 진폭계산에 유용하게 이용될 수 있는 u-게이지를 도입한다. 이를 이용해 그동안 알려지지 않았던 8개의 입자의 산란 진폭을 계산해낼 수 있음을 보인다. 또한 ABJM 산란 진폭의 재귀관계를 이용하여 ABJM 부드러움 정리(soft theorem)를 유도하고, 이를 통해 8개의 입자 산란 진폭 결과를 확인한다. 두번째 부분에서는 고차원 초등각장이론의 이중성 확인에 관하여 논의한다. 구체적으로, 각각 10개의 기본표현 물질을 갖는 5차원 $Sp(2)$ 게이지 이론과 $SU(3)$ 게이지 이론 사이의 이중성을 확인한다. 이 두 이론의 UV 고정점은 5차원이 아닌 원에 감긴 6차원 초등각장이론으로 추측되어진다. 이 6차원 초등각장이론은 $Sp(1)$ 게이지 군과 10개의 기본표현 물질을 갖는다. 순간자는 이런 이중성에 매우 중요한 역할을 하는데, 끈이론은 이런 순간자를 연구하는데 매우 유용하다. 끈이론을 이용하여 위의 세가지 이론의 순간자 분배함수를 계산하고 이를 통해 이중성을 확인한다.

주요어: ABJM, 산란 진폭, 순간자, D-막, 지표

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